AD-770 281

TOWARD THE DEFINITION OF ESCAPE AND CAPTURE REGIONS FOR A TWO AIRCRAFT PURSUIT-EVASION GAME

P. H. Cawdery

Air Force Institute of Technology
Wright-Patterson Air Force Base, Ohio

June 1973

DISTRIBUTED BY:

NTIS

National Technical Information Service
U. S. DEPARTMENT OF COMMERCE
5285 Port Royal Road, Springfield Va. 22151

UNCLASSIFIED

Security Classification

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author) Air Force Institute of Technology (AU) Wright-Patterson AFB, Ohio 45433	2a. REPORT SECURITY CLASSIFICATION UNCLASSIFIED 2b. GROUP

3. REPORT TITLE

TOWARD THE DEFINITION OF ESCAPE AND CAPTURE REGIONS FOR A TWO AIRCRAFT PURSUIT-EVASION GAME

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)

AFIT Thesis

5. AUTHOR(S) (First name, middle initial, last name)

P. H. Cawdery
Flt/Lt RAF

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
June 1973	~~65~~ 78	4

8a. CONTRACT OR GRANT NO.	9a. ORIGINATOR'S REPORT NUMBER(S)
b. PROJECT NO.	GA/MC/73-4
c.	9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)
d.	

10. DISTRIBUTION STATEMENT

Approved for public release; distribution unlimited

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY
	Air Force Armament Development & Test Center (AFATL) Eglin Air Force Base, Florida

13. ABSTRACT

The motivation for this thesis originates in research currently being conducted at the USAF Armament Laboratory, Eglin AFB, Florida. These studies concern the performance of an F4-E aircraft in air-to-air combat; the weapon system considered is an infra-red, heat-seeking missile. The studies fall into two categories:

(a) Definition of those regions in the vicinity of a target aircraft which the attacker must penetrate in order to attain a probability of killing his opponent greater than zero.

(b) Definition of optimal strategies for the attacker to intercept and penetrate the high probability of kill (P_K) regions.

In all cases, the target aircraft is considered as passive and unaware of attack.

This paper makes the logical extension to the above research, and attempts to develop a method by which the capability of the attacker may be defined against an intelligent and evasive target. The primary objective is to obtain regions for both aircraft which define or enclose those points in the game state space from which the attacker can always penetrate to a given probability of kill. These regions are called "capture" regions; the converse, for the target, are "escape" regions.

The air-to-air combat encounter is considered as a free time, zero sum, perfect

DD FORM 1 NOV 65 1473

UNCLASSIFIED

Security Classification

13. ABSTRACT (Continued)

information differential game. The participants' dynamics are modelled upon an F4 type aircraft, the game state space is defined, and the P_K regions modelled mathematically. An original extension to classical differential game theory is then made by which it is shown that partitions of the game space into escape and capture regions can be made for the simple planar game models. These regions are separated by a boundary which will be called a "P_K barrier." Obviously, the extent of the region from which the attacker can "capture" a given P_K value is a measure of his capability against the evasive, fully-informed target.

The theoretical development is applied to two planar game models. Numerical methods are used to generate optimal trajectories by backward integration from admissable terminal conditions for the game. These trajectories are analyzed, and partitions, or P_K barriers, are shown to exist. Examples of the escape and capture regions are shown, within the limits of the graphical techniques currently available.

Two major conclusions are made. From the analytic viewpoint, the methods developed show that partitions of the game space are possible for this class of game. Refinement of these methods would realize the potential of this form of analysis in defining the capability of an attacking aircraft in a variety of air combat situations. In the practical sense, it is shown that the particular weapon system modelled here has severe limitations when employed against an intelligent enemy. Although the analysis was restricted to two-dimensional maneuver for both aircraft, it is felt that generalization . the methods to three dimensions would reinforce the two-dimensional conclusions.

ii

14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Differential Game						
Pursuit-Evasion						
Escape/Capture						

AD-770281

TOWARD THE DEFINITION OF ESCAPE AND CAPTURE REGIONS FOR A TWO AIRCRAFT PURSUIT-EVASION GAME

THESIS

GA/MC/73-4

F. H. Cawdery
Flt/Lt RAF

i

GA/MC/73-4

TOWARD THE DEFINITION OF ESCAPE AND CAPTURE REGIONS FOR A TWO AIRCRAFT PURSUIT-EVASION GAME

THESIS

Presented to the Faculty of the School of Engineering of

the Air Force Institute of Technology

Air University

in Partial Fulfillment of the

Requirements for the Degree of

Master of Science

by

P. H. Cawdery, BSc

Flight Lieutenant, RAF

Graduate Astronautics

June 1973

id

Preface

This work is the outcome of my efforts to analyze the capability of a fighter aircraft equipped with a heat-seeking, air-to-air missile in pursuit of an intelligent and evasive target aircraft. Based upon classical differential game theory, an original method of analysis is developed by which escape and capture regions for each aircraft may be defined. The capability of the pursuing aircraft can then be measured by the extent of the regions from which it can achieve a specified probability of killing the target at missile launch. The analysis, and its application to planar game models, represents the initial steps toward complete definition of escape and capture regions for this class of differential games.

The thesis received its original inspiration from studies being conducted at the United States Air Force Armament Laboratory, Eglin AFB, Florida. These studies were basically concerned with the capability of an F4-E in air-to-air combat against a passive target aircraft, and this work makes the logical extension.

In concluding the paper, I would like to mention the debts I incurred during its development. My advisor, Professor Gerald M. Anderson of the Air Force Institute of Technology contributed much in originality and advice. My thanks also go to the staff of the Missile Systems and Analysis Division of the USAF Armament Laboratory at Eglin, with whom I had the pleasure of working for some eleven weeks during the crucial phases of the research.

Contents

Contents

List of Figures

Abstract

The motivation for this thesis originates in research currently being conducted at the USAF Armament Laboratory, Eglin AFB, Florida. These studies concern the performance of an F4-E aircraft in air-to-air combat; the weapon system considered is an infra-red, heat-seeking missile. The studies fall into two categories:

(a) Definition of those regions in the vicinity of a target aircraft which the attacker must penetrate in order to attain a probability of killing his opponent greater than zero.

(b) Definition of optimal strategies for the attacker to intercept and penetrate the high probability of kill (P_K) regions.

In all cases, the target aircraft is considered as passive and unaware of attack.

This paper makes the logical extension to the above research, and attempts to develop a method by which the capability of the attacker may be defined against an intelligent and evasive target. The primary objective is to obtain regions for both aircraft which define or enclose those points in the game state space from which the attacker can always penetrate to a given probability of kill. These regions are called "capture" regions; the converse, for the target, are "escape" regions.

The air-to-air combat encounter is considered as a free time, zero sum, perfect information differential game. The participants' dynamics are modelled upon an F4 type aircraft, the game state space is defined, and the P_K regions modelled mathematically. An original extension to classical differential game theory is then made by which it is shown that partitions of the game space into escape and capture regions can be made for the simple planar game models. These regions are separated

by a boundary which will be called a "P_K barrier." Obviously, the extent of the region from which the attacker can "capture" a given P_K value is a measure of his capability against the evasive, fully-informed target.

The theoretical development is applied to two planar game models. Numerical methods are used to generate optimal trajectories by backward integration from admissable terminal conditions for the game. These trajectories are analyzed, and partitions, or P_K barriers, are shown to exist. Examples of the escape and capture regions are shown, within the limits of the graphical techniques currently available.

Two major conclusions are made. From the analytic viewpoint, the methods developed show that partitions of the game space are possible for this class of game. Refinement of these methods would realize the potential of this form of analysis in defining the capability of an attacking aircraft in a variety of air combat situations. In the practical sense, it is shown that the particular weapon system modelled here has severe limitations when employed against an intelligent enemy. Although the analysis was restricted to two-dimensional maneuver for both aircraft, it is felt that generalization of the methods to three dimensions would reinforce the two-dimensional conclusions.

I. Introduction

The two aircraft pursuit-evasion problem has received a great deal of attention from researchers in recent years. The basic objective of the research has been to find analytical or numerical means of evaluating the effectiveness of one aircraft in competition with another. Many different situations arise dependent chiefly upon the relative capabilities of pursuer and evader, and the types of airborne weapons employed.

Much of the previous effort has been directed towards obtaining optimal strategies for pursuer and evader. These strategies arise directly from solutions to a given problem using differential game theory. Open, (and, in some cases closed) loop control laws can be obtained which define optimal play for each aircraft. However, this approach makes the assumption that the pursuer is initially in a position to force termination. Alternatively, optimal strategies can be obtained for an aircraft attacking a passive target. The major failing in this case is that the resulting strategies are non-optimal against a target which deviates from its specified trajectory.

This thesis approaches the pursuit-evasion game in what is believed to be an original manner. The problem is considered as an extension of the Isaacs "game of kind" (Ref (3)). Termination of the game is defined when the pursuing aircraft reaches a specified value (payoff) at the terminal time (t_f) and is boresighted on the evader. If the pursuer can attain a higher payoff than that specified and maintain boresight, capture is said to occur. If the evader prevents the pursuer reaching the specified terminal conditions, escape is said to occur. Using this extended concept from differential game theory, the intent of this thesis is to develop a method of partitioning the game state space into escape

and capture regions. Definition of such regions for a given aircraft/weapon system could have a significant impact on tactics and design.

The payoff, or value, of the game is defined in terms of the pursuer's probability of kill (P_K) if a missile is launched at t_f. Contours of constant P_K are modelled, and the pursuer then attempts to reach as high a value of P_K as possible (with boresight) before firing.

Two constant altitude models of the pursuit-evasion game are developed, in the first of which both aircraft are constrained to constant velocity; the second model permits variable velocity dependent upon thrust and drag forces. Chapter II introduces the game models, while Chapter III presents the theoretical aspects of differential game theory employed in the solution approach. Chapter IV then develops a method of specifying the admissible end points for the free time differential game.

Having specified a set of admissible end points, numerical backward integration is used to obtain optimal trajectories. Chapter V presents the solutions for the constant velocity model, and discusses the means by which partitions of the game space can be made. Some aspects of the variable velocity model are considered in Chapter VI, and Chapter VII presents a general discussion of the results and their possible implications. Conclusions and recommendations are contained in Chapter VIII.

It is felt that this research makes several contributions to the study of differential games and pursuit-evasion problems. As far as is known, no previous attempts have been made to define escape and capture regions for this class of game. Thus, the solution method employed has some interesting elements of originality. Secondly, successful definition of escape and capture regions allows a unique method of comparing

the capabilities of different aircraft, or of different weapon systems. Although the approach needs much refinement, it has great potential in the design and tactical application of airborne weapon systems.

II. Statement of the Problem

Origins of the Problem

The problem considered here owes its origin to studies in progress at the USAF Armament Laboratory. These studies concern the performance of an F4 type aircraft, equipped with a short-range missile, when employed against a passive target aircraft. Two major areas are considered. Firstly, the definition of those regions in the vicinity of the target where the attacker's probability of kill (P_K) is greater than zero. Secondly, the investigation of optimal strategies for the attacker which permit him to penetrate the P_K regions.

The question naturally arises as to the attacker's capability when the target assumes an intelligent, and hence evasive, role. This is the problem addressed by this thesis.

Thesis Objectives

The primary objective is to develop a method by which the state space for a two aircraft pursuit-evasion game may be partitioned into escape and capture regions. A capture region is defined as that region containing all starting points for the game from which the pursuer can exceed a specified $P_K(t_f)$ with boresight. An escape region is the converse. The boundary which separates these regions will be defined as a "P_K barrier."

The secondary objective is to consider the dependence of these regions on the various parameters of the game, and what impact definition of the regions may have on tactics and design.

Game Scenario

The game scenario has the following essential ingredients:

(a) The dynamics of both aircraft are based upon the F4 with variable

individual characteristics.

(b) The pursuer is equipped with a short-range, heat-seeking, air-to-air missile. The pursuer must be boresighted on the evader before firing the missile in order to permit seeker lock-on.

(c) Regions where the pursuer has a $P_K > 0$ are defined, which the pursuer attempts to penetrate before firing.

(d) Maneuvering is limited to the horizontal plane.

It is recognized that the restriction at (d) above limits direct practical application. However, assuming planar maneuvers reduces the dimensions of the game space, and hence of the solution trajectories, which in turn allows a more simple approach. So great is the problem of dimension, that the further restriction of constant velocity is also imposed in the initial analysis.

Aircraft Model

The following assumptions are made in respect of the aircraft models:

(a) A flat earth with constant gravitational acceleration.

(b) The aircraft are point masses.

(c) Thrust is a linear function of velocity, and is tangent to the aircraft flight path.

(d) Aircraft weight is constant.

(e) Aircraft load factor (n) is governed by

$$n \leq 5 \tag{2-1}$$

(f) The load factor for lift limited flight is a linear function of velocity.

(g) The drag polar can be represented as

$$C_D = C_{D_0} + k_1 C_L^2 + k_2 C_L^4 \tag{2-2}$$

The aircraft model is presented in detail in Appendix A.

Target Set Model

The target set is defined as that region in the vicinity of the target aircraft where the attacker's P_K is greater than zero. The model used in this approach is shown in Figure 1 overleaf. The target set is modeled as elliptical contours of constant P_K. The point 5000 ft directly to the rear of the evader has a P_K of 1.0. The zero P_K contour is an ellipse of semi-minor axis (a) 3000 ft and semi-major axis (b) 6000 ft centered at the $P_K = 1.0$ point. Thus, for example, the concentric ellipse with a = 2000 ft and b = 4000 ft represents a P_K of 5/9.

State Equations of Motion

The dynamic equations are written in a non-rotating frame fixed on the evader, as shown in Figure 2 (Page 8). The state vector is

$x_1 \triangleq$ Pursuer's position in x_1-direction

$x_2 \triangleq$ Pursuer's position in x_2-direction

$x_3 \triangleq$ Pursuer's heading

$x_4 \triangleq$ Evader's heading

$x_5 \triangleq$ Pursuer's velocity

$x_6 \triangleq$ Evader's velocity

Using the aircraft model developed in Appendix A, the state equations of motion may then be written

$$\begin{aligned}
\dot{x}_1 &= x_5\cos x_3 - x_6\cos x_4 \\
\dot{x}_2 &= x_5\sin x_3 - x_6\sin x_4 \\
\dot{x}_3 &= \frac{gu_p}{x_5} \\
\dot{x}_4 &= \frac{gu_e}{x_6}
\end{aligned} \qquad (2\text{-}3)$$

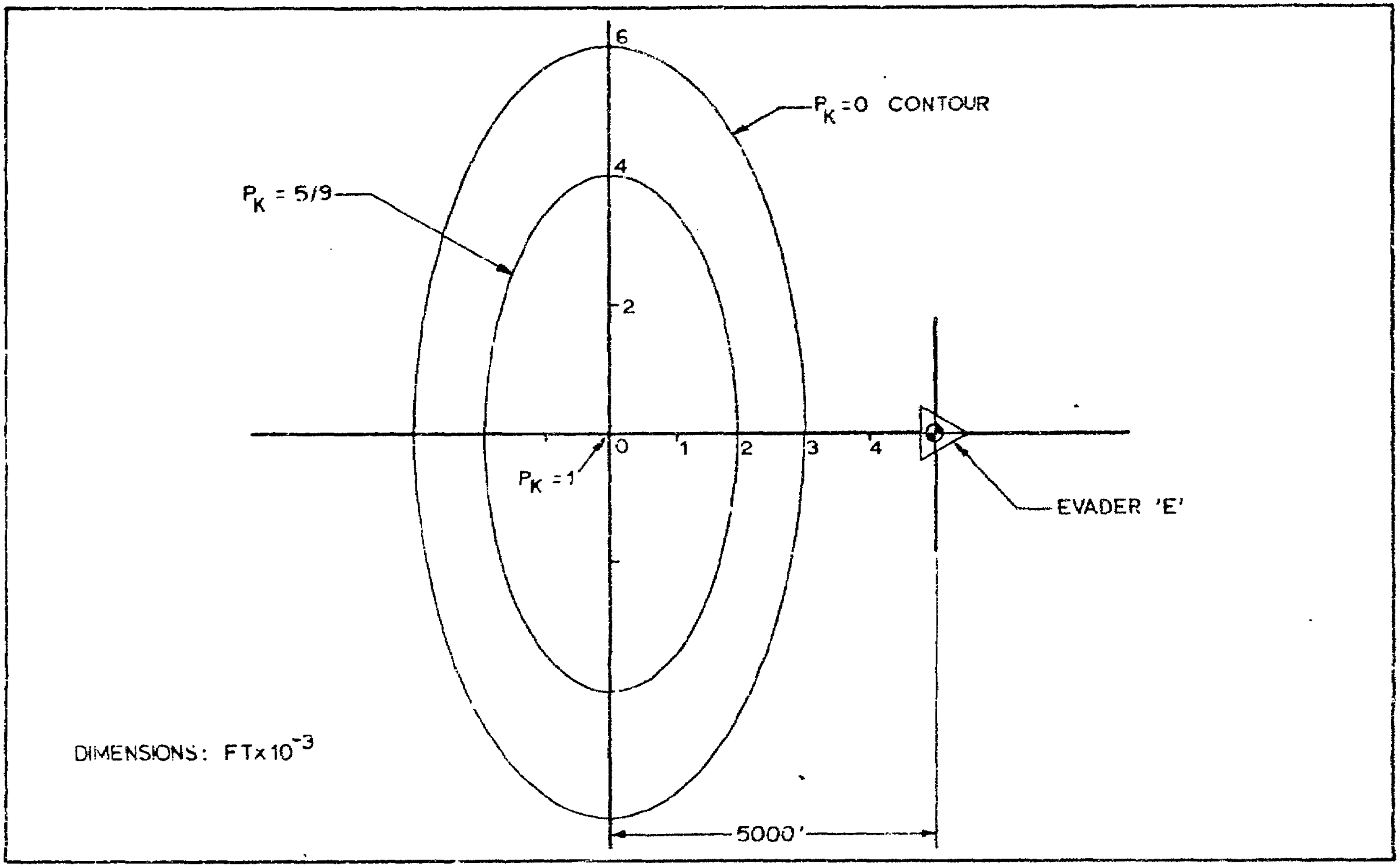

Fig. 1. Target Set Model

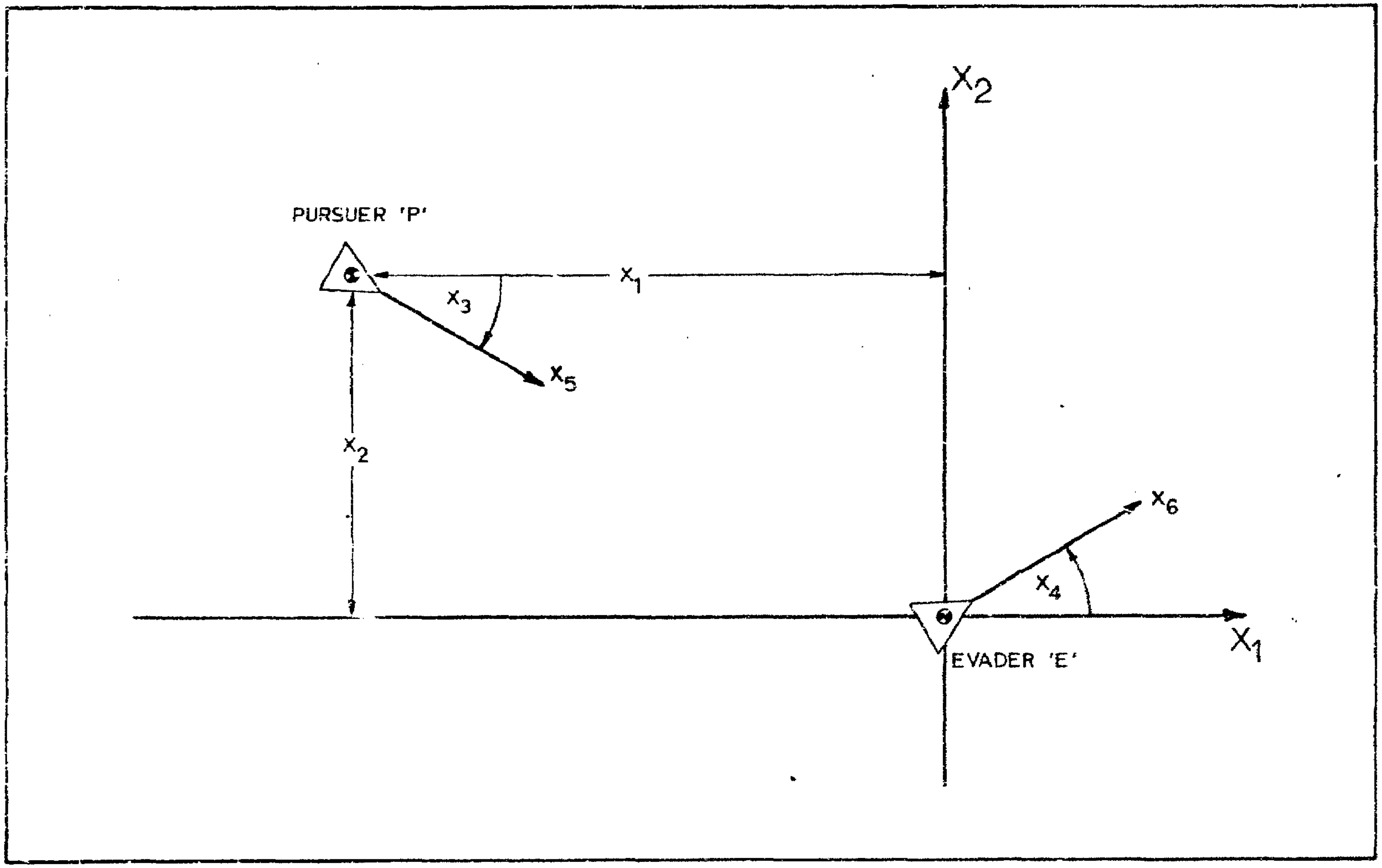

Fig. 2 State Variables for Game Dynamic Model

$$\dot{x}_5 = A_1 + A_2x_5 + A_3x_5{}^2 + \frac{A_4}{x_5{}^2}(u_p{}^2 + 1) + \frac{A_5}{x_5{}^2}(u_p{}^2 + 1)^2$$

$$\dot{x}_6 = A_1 + A_2x_6 + A_3x_6 + \frac{A_4}{x_6{}^2}(u_e{}^2 + 1) + \frac{A_5}{x_6{}^2}(u_p{}^2 + 1)^2$$

where u_p and u_e are the pursuer's and evader's controls respectively.

If the aircraft velocities are constant the Eq (2-3) becomes

$$\dot{x}_1 = V_p\cos x_3 - V_e\cos x_4$$

$$\dot{x}_2 = V_p\sin x_3 - V_e\sin x_4$$

$$\dot{x}_3 = \frac{gu_p}{V_p} \tag{2-4}$$

$$\dot{x}_4 = \frac{gu_e}{V_e}$$

where V_p and V_e are the pursuer and evader velocities respectively.

Target Set Equation

In the selected frame of motion, the target set equation is

$$P_K = 1 - \frac{(5000 + x_1\cos x_4 + x_2\sin x_4)^2}{a^2} - \frac{(-x_1\sin x_4 + x_2\cos x_4)^2}{K^2a^2} \tag{2-5}$$

where $a = 3000$ ft. The constant "K" defines the eccentricity of the elliptical contours. In all cases considered, K is set to a value of 2.0.

Bore-sight Condition

The bore-sight condition (χ) is placed on the pursuer to enable acquisition of the target by the missile seeker prior to firing. Mathematically stated it is

$$\chi(\underline{x}(t_f)) = \tan x_3 - \frac{x_2}{x_1} = 0 \qquad (2\text{-}6)$$

Summary of the Problem Formulation

The variable velocity model is formulated with a six-dimensional state vector; the constant velocity model with a four-dimensional state vector. Mathematically, the number of states could be reduced by one in each case, since a relative heading variable could replace individual pursuer and evader headings. However, there are mathematical advantages in solution with the equations in their stated form.

The models have the following major restrictions:

(a) Constraint to motion in a plane is unrealistic, since violent out-of-plane maneuvers can be expected in real air combat.,

(b) The dynamics of the variable velocity model are valid only for Mach No $\leq$ 0.9.

The restrictions certainly limit the practical reality of the results, but a simple model of reduced dimension offers, at least initially, a better opportunity of developing a solution method. Even in the cases considered capture/escape regions are 3-dimensional for the constant velocity model, and 5-dimensional for variable velocity. These dimensions cause severe practical difficulties in comprehension and presentation.

III. Differential Game Theory

The problem treated in this thesis is formulated as a free time, zero-sum, perfect information differential game, and the first requirement is to establish unique solutions for the game. The solution trajectories obtained must then be analyzed to determine whether the game state space, G, is partitioned into escape and capture regions. The purpose of this chapter is to define this class of games mathematically and to summarize those elements of differential game theory which are subsequently employed. The basis for this theory is contained in references (1), (2) and (3).

Class of Game

The objective of the differential game is to find

$$\min_{u} \max_{v} J = \phi\,(x(t_f)) \tag{3-1}$$

subject to the dynamic constraints

$$\dot{\underline{x}} = \underline{f}(x,u,v,t) \quad , \quad \underline{x}(t_o) = x_o \tag{3-2}$$

and the algebraic terminal constraint

$$\chi(\underline{x}(t_f)) = 0 \tag{3-3}$$

where $\underline{x}$ is the n-dimensional state vector, u is the pursuer's control and v is the evader's control. u and v may be subject to constraints.

The aim is to find the controls u* and v* such that

$$J(u^*,v) \leq J(u^*,v^*) \leq J(u,v^*) \tag{3-4}$$

If the pair (u*,v*) can be found, they constitute a saddle point solution of the game and J(u*,v*) is called the value of the game.

Necessary Conditions for a Solution

The existence of a solution is dependent upon the fact that

$$\min_{u} \max_{v} J(u,v) = \max_{v} \min_{u} J(u,v) \tag{3-5}$$

A necessary condition for a saddle point solution is that the Hamiltonian (H) defined as

$$H(\underline{x},\underline{\lambda};u,v,t) = \underline{\lambda}^T \underline{f} \tag{3-6}$$

must be minimized over the admissable set of u and maximized over the admissable set of v such that

$$H^* = \min_{u} \max_{v} H = \max_{v} \min_{u} H \tag{3-7}$$

$\underline{\lambda}$ is the n-dimensional costate vector and

$$\dot{\underline{\lambda}} = - H_{\underline{x}} \tag{3-8}$$

subject to the transversality conditions

$$\underline{\lambda}(t_f) = \phi_{\underline{x}}(t_f) + \nu\chi_{\underline{x}}(t_f) \tag{3-9}$$

$$H(t_f) = \phi_t(t_f) + \nu\chi_t(t_f) \tag{3-10}$$

where ν is an arbitrary constant multiplier.

Further, if t does not appear explicitly in Eq (3-6) then H is constant. An important outcome of this condition is that H, ϕ and χ are not functions of t in the problem considered and condition (3-10) may then be written

$$H(t) = H(t_f) = 0 \tag{3-11}$$

Eq (3-7) implies that the maximization and minimization of H commute, which is not true in general. It is true, however, if H is separable into two functions, one of which is independent of v, the other independent of u. For the problems considered in this thesis, f and hence H is

separable. This insures that the minimizing u and maximizing v provide a saddle point in H at each point of the optimal path.

Singular Controls

When the controls u and v appear linearly in Eq (3-6), then the possibility of solution arcs with singular controls exists. In the case of the constant velocity model, this situation arises and singular arcs occur. The necessary and junction conditions for such arcs are discussed in Appendix B.

Games of Kind

Isaacs (Ref (3)) introduces the concepts of the "game of kind" and the "game of degree." In the game of kind, the primary concern is whether or not termination (as defined for the game) occurs. This contrasts with the game of degree where termination is assumed to occur, and the player's objectives are to hasten or delay termination, or to minimax a continuous payoff. The differential game is considered here in the context of a game of kind, the object being to determine whether termination occurs from a given set of starting conditions.

The Barrier Concept. A game of kind in the game space G is assumed, with a terminal surface X specified. P attempts to penetrate X, while E attempts to prevent penetration. There are three possible outcomes

(a) P penetrates X (capture)

(b) P does not penetrate X (escape)

(c) P just reaches X, but does not penetrate.

The "neutral" outcome at (c) is that of significance, since the trajectory resulting in (c) is, in a sense, the only one on which the strategies of P and E are decisive. That is, non-optimal play by E will

result in P penetrating X, while non-optimal play by P will result in E's escape.

The assumption now made is that G contains starting points which result in either capture or escape. Generally, these points will fall into regions which are separated by a surface consisting of those starting points for which the outcome is neutral. This surface is a "barrier", an example of which may be seen in the "Homicidal Chauffeur Game" analyzed by Isaacs (Ref (3)). The barriers give vital information on the relative capabilities of P and E, and, if they are shown to enclose entirely some portion of G, then the space is automatically divided into escape and capture regions.

Construction of the Barrier. This section summarizes Isaacs' work in Ref (3). The termination of the problem is assumed at t_f. A neutral outcome demands that, while P's path touches X, it does not penetrate. Physically, this is equivalent to P having a zero rate of penetration at t_f, or that the component of P's velocity normal to χ is zero. Mathematically this can be written

$$\chi_{\underline{x}}^T(t_f)\dot{\underline{x}}(t_f) = 0 \tag{3-12}$$

If a Lagrange multiplier $\underline{\alpha}$ is defined such that

$$H_B = \underline{\alpha}^T\dot{\underline{x}} = \underline{\alpha}^T\underline{f} \tag{3-13}$$

and

$$\underline{\alpha}(t_f) = \chi_{\underline{x}}(t_f) \tag{3-14}$$

then Eq (3-12) can be written

$$H_B(t_f) = 0 \tag{3-15}$$

where H_B is defined as the "Barrier Hamiltonian."

A point on χ where Eq (3-15) is satisfied is called the "Boundary

of the Useable Part" (BUP), since it separates those portions of χ where P has a positive rate of penetration (the Useable Part) and a negative rate of penetration (Non-useable Part).

There is an obvious equivalence between Eqs (3-6) and (3-13). In fact, in the absence of J, they are identical. Thus, a backward integration of Eqs (3-2) and (3-8) subject to solution of Eq (3-15) and the necessary conditions yields the classical Isaacs barrier.

The Dispersal Surface

The concept of the dispersal surface (DS) (Ref (3)) is one which has an important bearing on the analysis of a differential game solution. Generally, there are two aspects to the given game solution. One is solution "in the small" where, assuming $\underline{x}(t_f)$ and $\underline{\lambda}(t_f)$ are known, backward integration will yield a solution trajectory. However, there exist singular surfaces in the G-space which delineate regions of different behavior of the dynamic equations. Ascertaining these surfaces is termed the solution "in the large." The implication of the existence of singular surfaces is that solutions "in the small" may be invalidated because of the presence of these surfaces.

A differential game is assumed, where optimal trajectories have been obtained by backward integration of the dynamic equations. Suppose also that the paths obtained fall into 2 classes and that paths from each class intersect as suggested by Figure 3. If, at the point of intersection, the states are identical, and the values (payoffs) for each path are equal, then the intersection is a point on a DS. The locus of such points is the DS. Only the paths from C to the DS are retained.

Points on the DS may also be considered as confronting E with a dilemma in choice of strategy, either choice resulting in an equal value

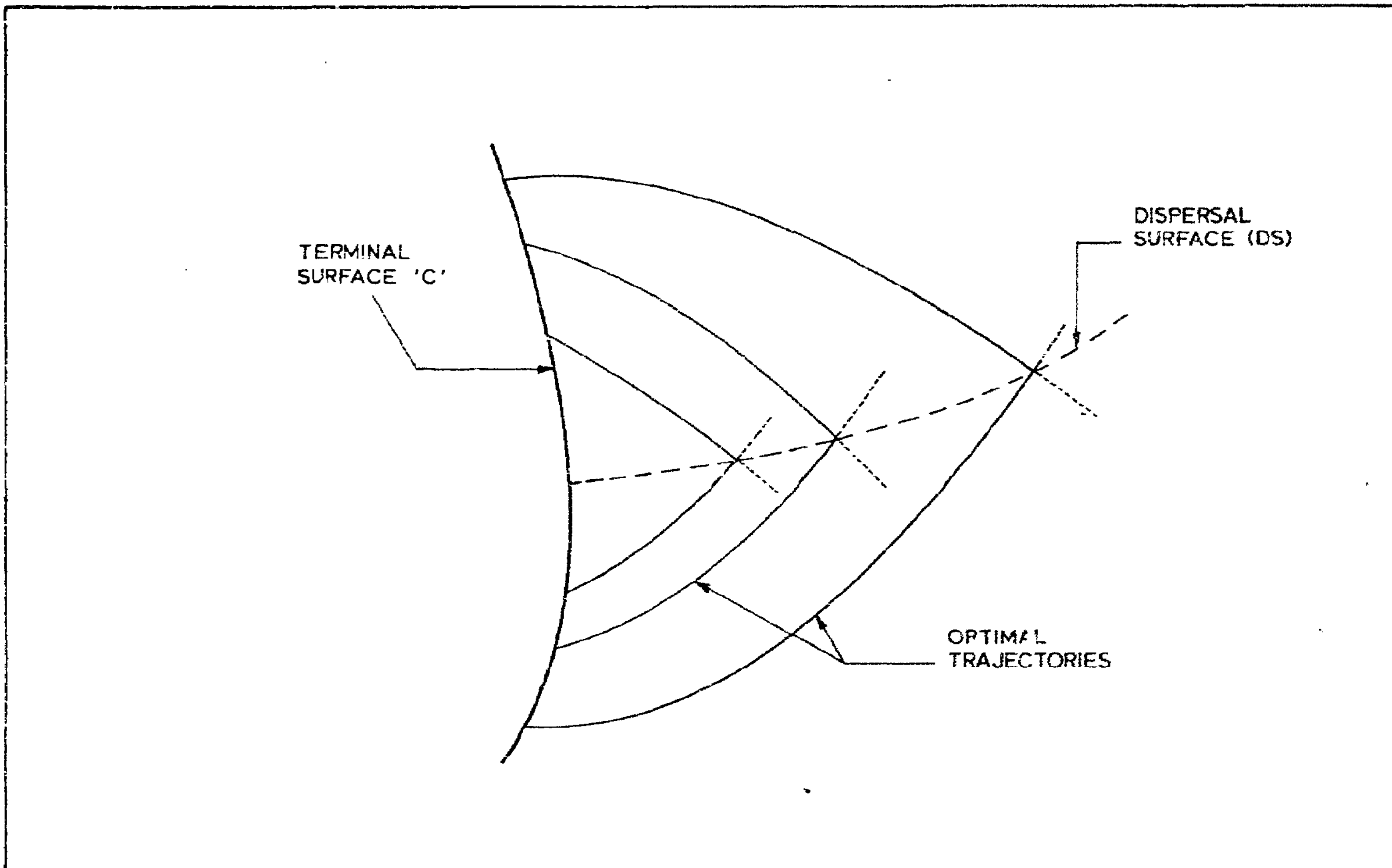

Fig. 3. The Dispersal Surface Concept

at termination. The importance of the DS is that it provides a method of terminating trajectories, and in a given problem may well provide the key to partitioning G.

The Solution Method

The major observation from the preceding sections arises from the comparison of Eqs (3-6) and (3-13). The multipliers $\underline{\lambda}$ and $\underline{\alpha}$ are not equivalent since $\underline{\alpha}(t_f)$ contains no influencing term in $\phi_{\underline{x}}$ as does $\underline{\lambda}(t_f)$ in Eq (3-9). Thus, in effect, an entirely new problem is posed. Not only does P have to maneuver so as to attain boresight, but has also to penetrate a region where he has a $P_K > 0$.

The terminal conditions obtained are not analogous to the BUP, which is principally dependent on a zero rate of penetration of a specified terminal surface. The terminal states in the present game are a combination of both the ability of P to reach C, and to satisfy the boresight condition. These two constraints together render the classical approach inapplicable. Hence barrier trajectories of the type developed by Isaacs do not exist for this problem, and the development of P_K barriers must then be accomplished by analysis of the solution trajectories.

The approach is based on the foregoing observations on classical theory. A particular P_K contour is specified as the terminal objective, and an admissable set of terminal conditions determined. Optimal paths can then be generated by backward integration, and analyzed to obtain the P_K barriers.

It is interesting to note that a BUP in the Isaacs sense would exist for this problem if the boresight condition (Eq (3-3)) above were considered. This classical BUP is discussed briefly in Chapter IV.

IV. Determination of Admissable Terminal Conditions

One method approaching the solution of the pursuit-evasion problem defined in this thesis is to use a backward integration technique. This implies complete specification of an admissable terminal point for the game. The purpose of this chapter is to develop a method whereby admissable terminal conditions may be specified.

Mathematical Formulation

Constant Velocity Model. Based upon the game model presented in Chapter II, the dynamic equations are

$$\begin{aligned} \dot{x}_1 &= V_p \cos x_3 - V_e \cos x_4 \\ \dot{x}_2 &= V_p \sin x_3 - V_e \sin x_4 \\ \dot{x}_3 &= \frac{g}{V_p} u_p \\ \dot{x}_4 &= \frac{g}{V_e} u_e \end{aligned} \tag{4-1}$$

$$\begin{aligned} \dot{\lambda}_1 &= 0 \\ \dot{\lambda}_2 &= 0 \\ \dot{\lambda}_3 &= V_p(\lambda_1 \sin x_3 - \lambda_2 \cos x_3) \\ \dot{\lambda}_4 &= V_e(-\lambda_1 \sin x_4 + \lambda_2 \cos x_4) \end{aligned} \tag{4-2}$$

The terminal boresight constraint is

$$\chi[\underline{x}(t_f)] = (\tan x_3 - \frac{x_2}{x_1})\Big|_{t_f} = 0 \tag{4-3}$$

The objective function is

$$J = 1 - P_K(t_f) \tag{4-4}$$

where

$$P_K = 1 - \frac{p^2}{a^2} - \frac{q^2}{4a^2} \qquad (4\text{-}5)$$

so that

$$\phi[\underline{x}(t_f)] = \left. \frac{p^2}{a^2} + \frac{q^2}{4a^2} \right|_{t_f} \qquad (4\text{-}6)$$

where

$$p = 5000 + x_1 \cos x_4 + x_2 \sin x_4 \qquad (4\text{-}7)$$

$$q = -x_1 \sin x_4 + x_2 \cos x_4 \qquad (4\text{-}8)$$

and 'a' is the semi-minor axis of a given P_K contour such that

$$P_K = 1 - \frac{a^2}{3(10^6)} \qquad (4\text{-}9)$$

The evader's heading, $x_4(t_f)$ can arbitrarily be specified as zero, since only relative heading is of importance. Then, applying condition (3-9)

$$\underline{\lambda}(t_f) = \begin{matrix} \frac{2}{a^2}(x_1 + 5000) + \nu \frac{x_2}{x_1^2} \\ \frac{x_2}{2a^2} - \frac{\nu}{x_1} \\ \nu \sec^2 x_3 \\ (3x_1 + 20{,}000) \frac{x_2}{2a^2} \end{matrix} \qquad (4\text{-}10)$$

and hence

$$H(t_f) = \Big[\frac{2}{a^2}(x_1 + 5000) + \nu \frac{x_2}{x_1^2} (V_p \cos x_3 - V_e) + \left(\frac{x_2}{2a^2} - \frac{\nu}{x_1} \right) V_p \sin x_3 + \nu \sec^2 x_3 \frac{g}{V_p} u_p + (3x_1 + 20{,}000) \frac{x_2}{2a^2} \frac{g}{V_e} u_e \Big] \Big|_{t_f} \qquad (4\text{-}11)$$

Now, condition (3-11) requires that $H(t_f) = 0$, so that setting Eq (4-11) equal to zero will yield a set of admissable end points for given V_p, V_e, ν, a, u_p and u_e. V_p and V_e may be arbitrarily specified, as may a, which defines the P_K contour of interest. The controls $u_p(t_f)$ and $u_e(t_f)$ are selected as those which minimize and maximize H respectively. x_2 and x_3 are eliminated from Eq (4-11) by using Eqs (4-3) and (4-5). Eq (4-11) is now reduced to a function of x_1 and ν. Parameterizing in ν, the equation may be solved numerically to yield the condition $H(t_f) = 0$ and specify $\underline{x}(t_f)$ and $\underline{\lambda}(t_f)$ for a given value of P_K and ν.

Variable Velocity Model. It is interesting to note that the end points determined in the preceding section are also those for the variable velocity game model. Since ϕ and χ are independent of V_p and V_e, $\lambda_5(t_f)$ and $\lambda_6(t_f)$ are zero. Thus equation (4-10) is independent of the equations defining $\dot{V}_p$ and $\dot{V}_e$ and its solution yields valid end points for both game models.

The Classical BUP

A classical BUP in the Isaacs sense can be shown to exist by considering only the boresight condition in Eq (4-3). Applying conditions (3-13) through (3-15) yields

$$x_2 x_5 \cos x_3 - x_2 x_6 - x_1 x_5 \sin x_3 + x_1^2 \sec^2 x_3 \frac{g u_p}{x_5} = 0 \qquad (4\text{-}12)$$

Eliminating x_3 from (4-12) using Eq (4-3) gives

$$x_1^2 + x_2^2 - x_2 \left(\frac{x_5 x_6}{g u_p} \right) = 0 \qquad (4\text{-}13)$$

It can be seen that Eq (4-13) is the equation of a circular BUP centered on the x_2-axis, and which is dependent upon u_p, the pursuer's control (or available rate of turn). Physically, the circle encloses

those points in the state space where the pursuer cannot hold boresight on E because of his turn rate limitation.

This BUP does not have any great practical significance in the problem considered, since the P_K regions are not intersected by the regions defined by Eq (4-13) for the game situations considered in this paper. The chief reason for making the foregoing observations is to provide the contrast between the classical analysis and the solution method developed here.

The Terminal Conditions

A typical set of solutions satisfying the terminal conditions are shown in figures 4 and 5 overleaf for V_p of 850 ft/sec and V_e of 780 ft/sec. Each value of ν affords one end point for any given value of P_K. Hence, each value of ν results in a locus of end points which is symmetrical about the x_1-axis. Only the solutions for the positive values of x_2 are shown. Several observations can now be made about the admissable end points and their effects on solution trajectories.

End Point Envelope. Figures 4 and 5 show that the solution end points are contained in an "envelope" bounded by the locus of end points for $\nu = 0$. This is typical of the solutions for any values of V_p and V_e at the terminal time, provided $V_p > V_e$. Numerically, however, Eq (4-11) in certain cases gives rise to end points not contained in the $\nu = 0$ envelope. These end points are considered invalid, since it is readily shown that P, having once attained boresight, can hold boresight with increasing P_K until the $\nu = 0$ locus is reached. In essence, the dynamic equations are integrated forward in time. E's control is optimal and P's control is that required to hold boresight on E; the P_K is shown to increase until the $\nu = 0$ locus is reached. The spurious end points are

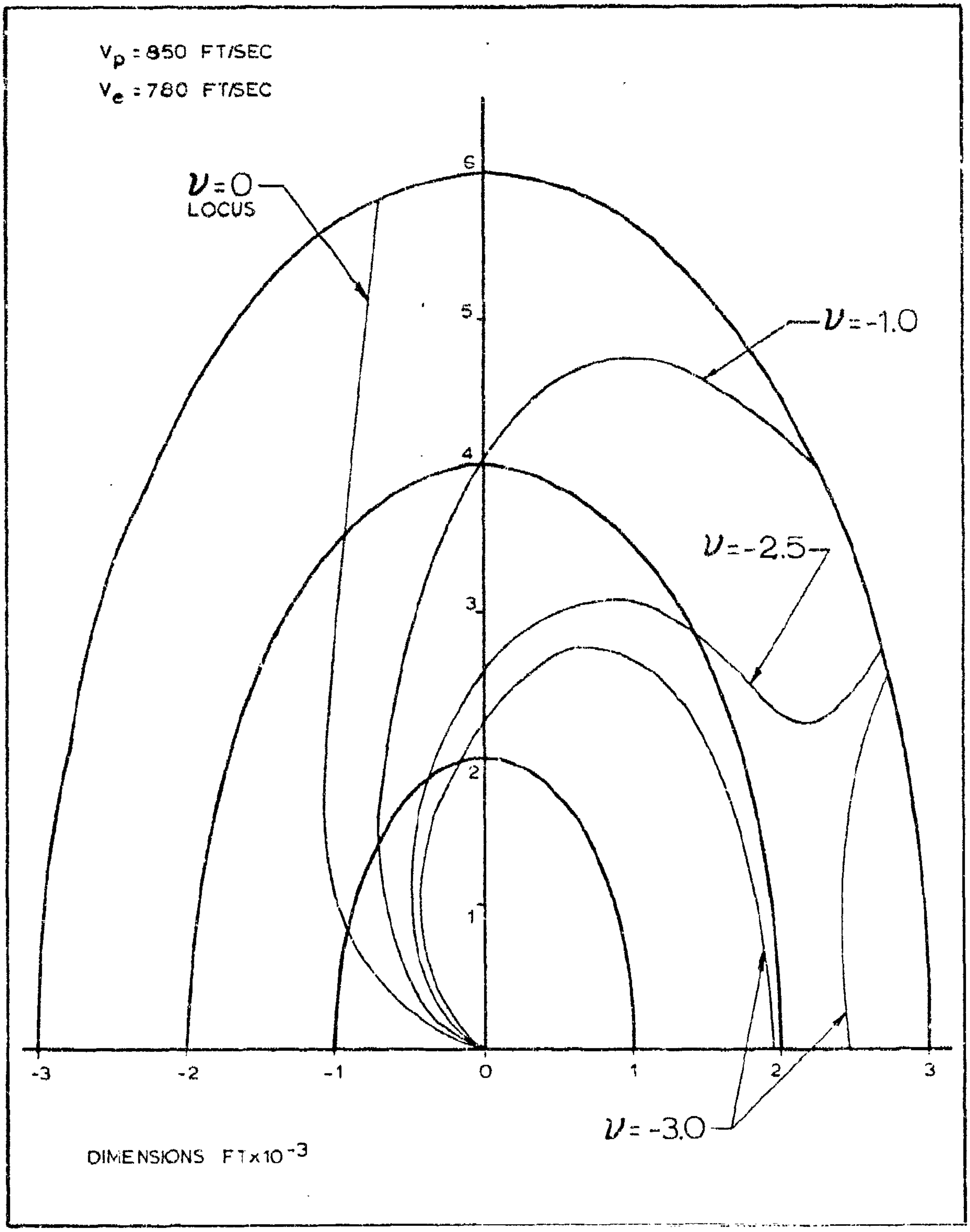

Fig. 4. Loci of Admissable End Points (ν < 0)

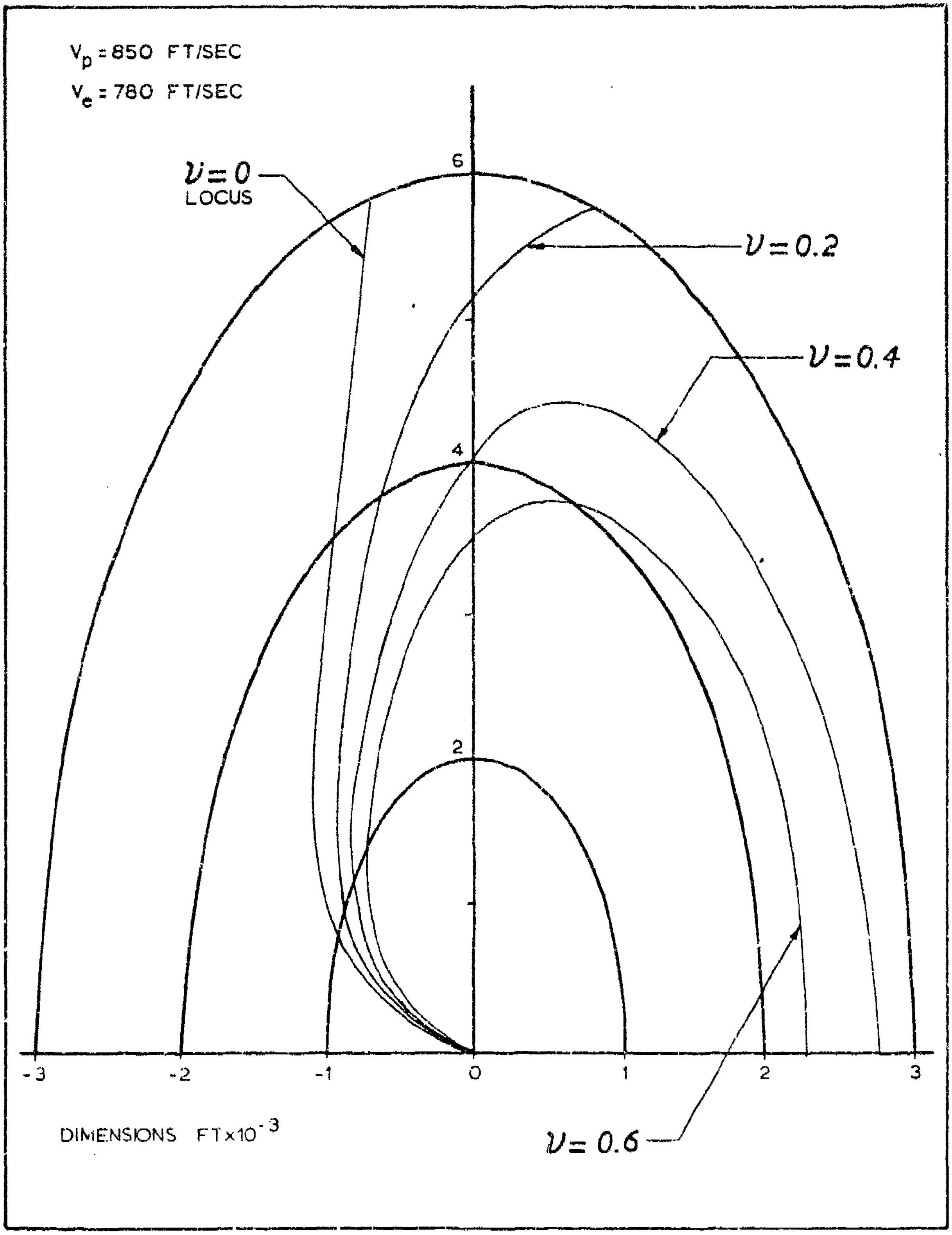

Fig. 5. Loci of Admissable End Points (ν > 0)

discarded on the basis of this reasoning.

Evader Control (u_e). The evader's control at the final time is influenced by $\lambda_4(t_f)$. Considering only the half plane for $x_2 > 0$, the value of $u_e{}^*(t_f)$ is that which maximizes $H(t_f)$. Now, from Eq (4-10)

$$\lambda_4(t_f) = (3x_1 + 20{,}000)\ \frac{x_2}{2a^2} \tag{4-14}$$

and the term in H containing $\dot{u}_e$ is

$$\lambda_4(t_f)\dot{x}_4(t_f) = (3x_1 + 20{,}000)\ \frac{x_2}{2a^2} \cdot \frac{g}{V_e}\, u_e \tag{4-15}$$

Thus

$$u_e{}^*(t_f) = u_{emax} \mathrm{Sgn}(\lambda_4(t_f)) \tag{4-16}$$

Pursuer Control (u_p). By similar reasoning to the previous subsection, it can be shown that

$$u_p{}^*(t_f) = -\ u_{pmax} \mathrm{Sgn}(\lambda_3(t_f)) \tag{4-17}$$

Thus $u_p{}^*(t_f)$ is entirely dependent on the sign of ν.

Summary

The solution method developed in this chapter permits the specification of terminal conditions for any given set of problem parameters. Numerical backward integration techniques may now be used to obtain optimal trajectories from any of the given terminal conditions.

V. Analysis of the Constant Velocity Model

The admissable sets of terminal conditions for the pursuit-evasion game can now be specified using the method developed in Chapter IV. The purpose of this chapter is to consider the generation of P_K barriers for the constant velocity model by analyzing optimal trajectories obtained by backward integration.

Mathematical Statement

The state equations of motion are

$$\begin{aligned} \dot{x}_1 &= V_p \cos x_3 - V_e \cos x_4 \\ \dot{x}_2 &= V_p \sin x_3 - V_e \sin x_4 \\ \dot{x}_3 &= \frac{g}{V_p} u_p \\ \dot{x}_4 &= \frac{g}{V_e} u_e \end{aligned} \tag{5-1}$$

The costate differential equations are

$$\begin{aligned} \dot{\lambda}_1 &= 0 \\ \dot{\lambda}_2 &= 0 \\ \dot{\lambda}_3 &= V_p(\lambda_1 \sin x_3 - \lambda_2 \cos x_3) \\ \dot{\lambda}_4 &= V_e(-\lambda_1 \sin x_4 + \lambda_2 \cos x_4) \end{aligned} \tag{5-2}$$

The objective function is

$$J(t_f) = \frac{p^2}{a^2} + \frac{q^2}{4a^2} \Big|_{t_f} \tag{5-3}$$

where p and q are defined by Eqs (4-7) and (4-8), and the terminal constraint at t_f is

$$\tan x_3 - \frac{x_2}{x_1} = 0 \tag{5-4}$$

The transversality conditions then yield $\underline{\lambda}(t_f)$ and $H(t_f)$ as defined by Eqs (4-10) and (4-11). The controls which minimize and maximize H are, respectively

$$u_p^* = -u_{pmax} \mathrm{Sgn}(\lambda_3) \tag{5-5}$$

$$u_e^* = u_{emax} \mathrm{Sgn}(\lambda_4) \tag{5-6}$$

The end point for a given set of parameters is determined, and backward integration of Eqs (5-1) and (5-2) then yields an optimal trajectory.

<u>Singular Arcs</u>. Application of the necessary conditions summarized in Appendix B give the following set of equations which must be satisfied if a singular arc is to exist for the pursuer:

$$\frac{\lambda_3 g}{V_p} = 0 \tag{5-7}$$

$$\rightarrow \quad \lambda_3 = 0 \tag{5-8}$$

$$\lambda_1 \sin x_3 - \lambda_2 \cos x_3 = 0 \tag{5-9}$$

$$\rightarrow \quad \tan x_3 = \frac{\lambda_2}{\lambda_1} \tag{5-10}$$

The optimal control on the singular arc is

$$u_p^* = 0 \tag{5-11}$$

<u>Control Sequences</u>. The controls yielded by Eqs (5-5) and (5-6) are physically equivalent to P and E flying maximum rate turns. Where singular arcs exist, Eq (5-11) indicates that P flies a straight, level "dash." These control sequences are typical for the case where the controls appear linearly in the state equations, since no continuous monotomic changes in control can occur. This is a direct result of applying the necessary conditions for a minimax solution.

<u>Trajectory Analysis</u>

There is no known precedent which would indicate a standard procedure

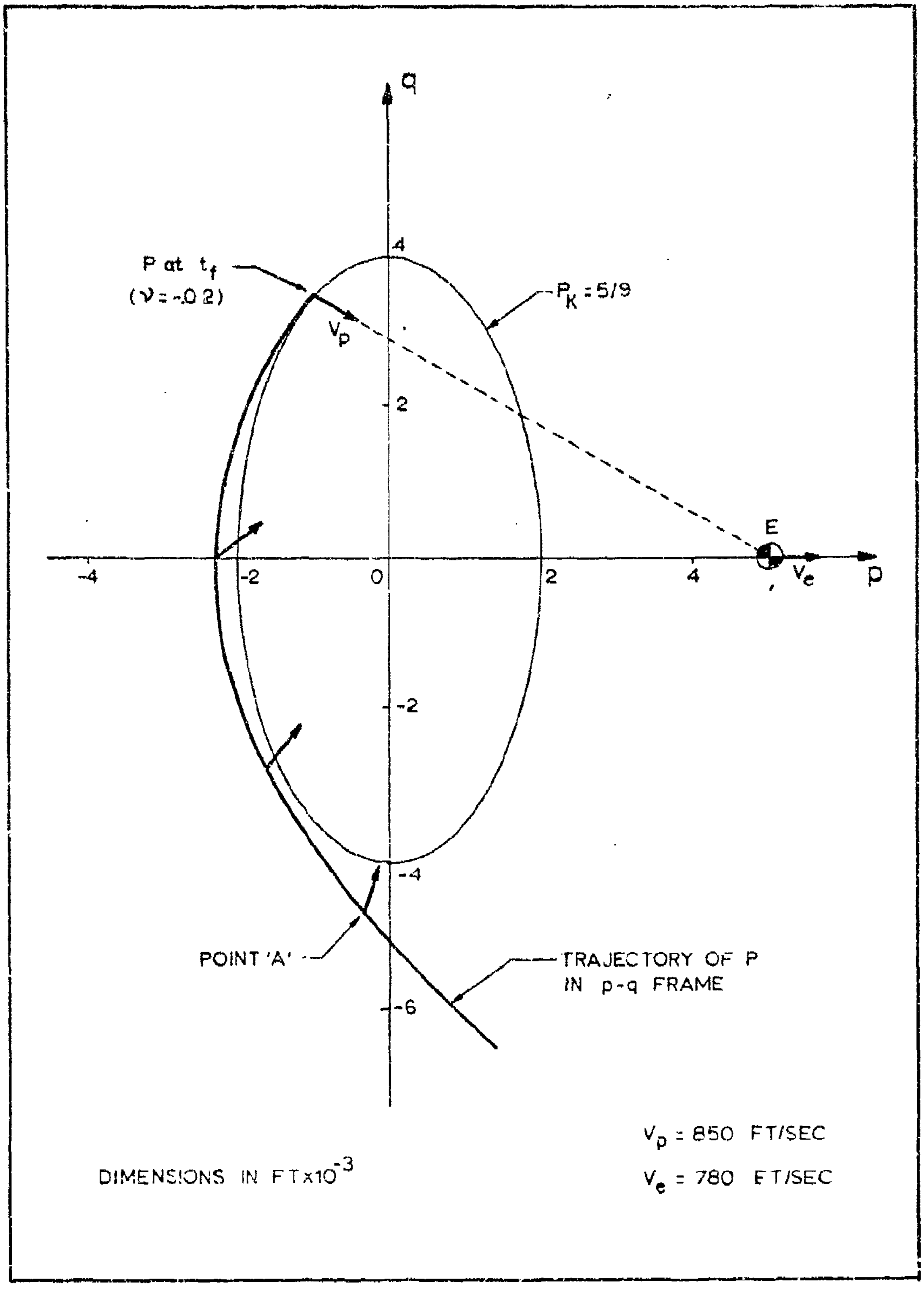

Fig. 6. Typical Optimal Trajectory

for determining partition trajectories. Thus, initially, a number of trajectories were obtained by numerical backward integration and their characteristics analyzed. These trajectories are considered in a coordinate frame relative to the evader which is centered at the $P_K = 1$ point and rotates with the evader. The transformation to the p-q coordinate frame is accomplished by applying Eqs (4-7) and (4-8).

In the subsequent analysis, the game parameters are

$$V_p = 850 \text{ ft/sec}$$

$$V_e = 780 \text{ ft/sec}$$

$$a = 2000 \text{ ft}$$

$$\nu \text{ variable}$$

Thus P's objective is to reach a region where $P_K \geq 5/9$ with bore-sight, and E attempts to prevent this termination. Computation of a possible end point is equivalent to assuming termination, and backward integration then yields the paths and strategies which would result in the given ending for a free time differential game.

Figure 6 (Page 27) shows a typical trajectory depicted in the p-q frame. P's heading at various points is shown by the arrows. The first consideration concerns the juxtaposition of P and E at the point labelled "A". Assume that the game were to commence at this point, and consider P's velocity in relation to the velocity of C (V_c) as shown in Figure 7 overleaf.

It is heuristically obvious that E's best strategy at this point would be a turn to the right, thus moving C away from P. V_c is a combination of V_e and the angular velocity due to E's turn rate ($\dot{x}_4$). The velocity V_c is greater than V_p, and hence E could prevent P from reaching C and terminating at the specified end point. The deduction made from this

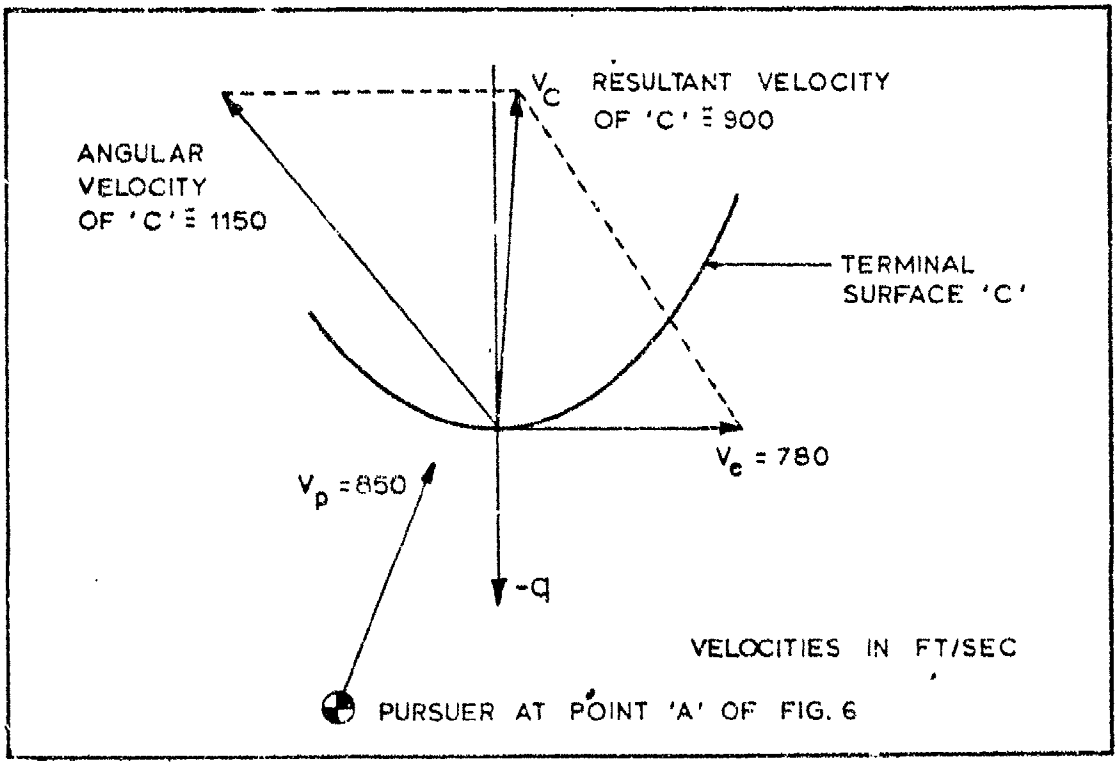

Figure 7. Comparison of V_p and V_e

reasoning is that, at some point, the trajectory in Figure 6 has crossed a singular surface; specifically, a dispersal surface. This invalidates a portion of the trajectory. The question which then arises is how to identify and locate the dispersal surface.

The Dispersal Surface. From Chapter III, the major requirements for a dispersal surface are:

(a) Intersection of paths of different classes.

(b) At intersection, the states must be identical.

(c) The payoff at termination is the same for each path.

Reconsidering Figure 6, it can be observed that for each end point on the positive q half-frame, there is a "mirror image" end point in the negative half-plane. Thus the end points are symmetrical about the p-axis,

as are the resulting optimal trajectories. A direct outcome of this observation is that symmetrical trajectories intersect on the p-axis. Hence the requirements at (a) and (c) are fulfilled, the payoff for the present example being a P_K of 5/9 at t_f.

Considering requirement (b), it is seen that, at the intersection of symmetrical trajectories, the positional states p and q are equal for coincidence. However, the relative heading state (x_3-x_4) is not the same for both trajectories. Although the magnitudes are equal, the directions are not. In only one case can coincidence of all three states be obtained for symmetrical trajectories and that occurs for $x_3=x_4$ (i.e., co-heading) at intersection on the p-axis. If two such symmetrical trajectories can be found, then at least one point on the dispersal surface can be identified.

The trajectory analysis reveals that the end points defined for $\nu<0$ result in trajectories that contain a switch of u_p from $+u_{pmax}$ to $-u_{pmax}$. This is a direct consequence of λ_3 passing through zero at some point in the trajectory. Further study shows that for some value of $\nu<0$ the conditions given by Eqs (5-8) through (5-10) are satisfied, and hence that a singular arc exists. In all the cases studied, this singular arc provides a means of obtaining the previously discussed point on the dispersal surface.

Singular Arc Trajectories. The satisfaction of the necessary conditions for a singular arc essentially devolves into a one parameter search over the values of $\nu<0$. Only one value of ν results in the satisfaction of the junction conditions for a singular arc for a given value of V_p, V_e and $P_K(t_f)$. Once the conditions (5-7) through (5-10) are reached, P's control is switched to the optimal value of zero (Eq (5-11)). Backward integration continues, and optimality of the resulting paths is maintained.

A further characteristic of the singular arc is that u_p may be

switched arbitrarily to $\pm u_{pmax}$ at any time on the arc while retaining optimality. The major importance of the singular arc is that it results in an infinite number of trajectories. By judicious manipulation of the time at which P's control is switched, once on the singular arc, a trajectory can be obtained which intersects the p axis in such a way that the relative heading (x_3-x_4) is zero at the point of intersection. By the previous reasoning, this point is on a dispersal surface, and the trajectories which produce it can be terminated at the point of intersection.

Partitions of the Game Space

The foregoing section discussed the construction of optimal trajectories and the presence of a dispersal surface. A method was also developed whereby at least one point on this surface may be defined. Two factors now require consideration:

(a) Whether the game space is in any way partitioned.

(b) If partition is shown to exist, how it can be diagrammatically represented.

The Existence of Partitions. Observation of the behavior of many optimal trajectories for $-3.0 \leq \nu \leq 0.4$ indicates that the point $(\underline{x}_{DS})$ identified as being on the dispersal surface bounds the values of p at which all other trajectories intersect the p-axis. The typical case is illustrated in Figure 8 overleaf by several example trajectories. From a heuristic standpoint, the above observation makes sense, since the point $\underline{x}_{DS}$ (Fig 8) includes the most advantageous heading for P on the negative p-axis. Hence, it could be concluded that, from a starting point at which $x_3-x_4 = 0$, P can achieve a $P_K(t_f) = 5/9$ from the least advantageous point on the negative p-axis, i.e., the point furthest from C. The question is whether this bound can be established mathematically.

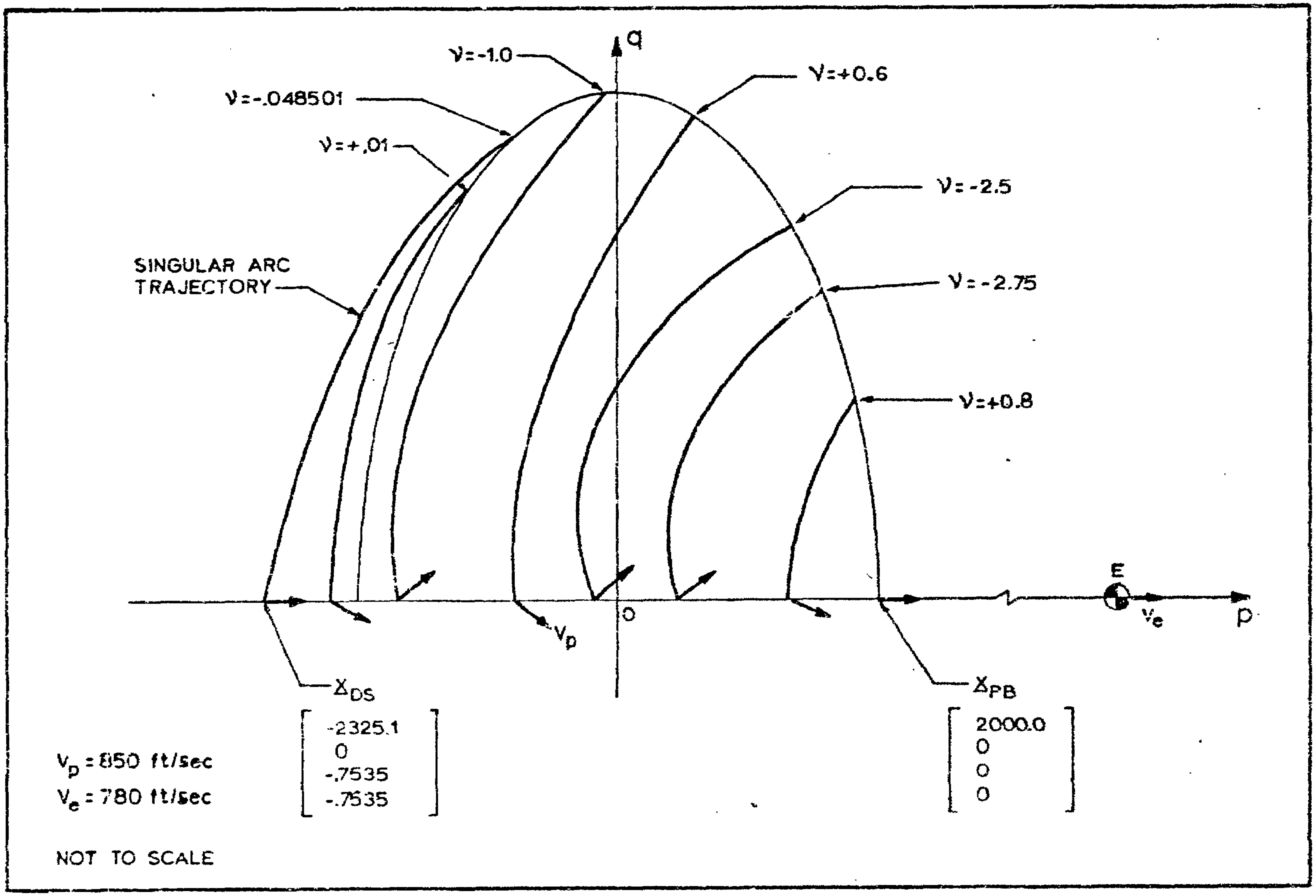

Fig. 8. Enclosure of Trajectories Intersecting the p-axis by $\underline{x}_{DS}$

<u>The Definition of Partion</u>. A linear perturbation analysis is developed in Appendix C which is used to show that $\underline{x}_{DS}$ includes the least value that p can assume for termination at $P_K(t_f) = 5/9$. Effectively, the costate values $\underline{\lambda}(t)$ can be regarded as influence coefficients on the payoff function $\bar{J}$, where

$$\bar{J} = \left. \phi(\underline{x}) + \nu X(\underline{x}) \right|_{t_f} \tag{5-12}$$

If $\underline{x}_{DS}$ is defined as the starting point for the game at time $t = t_o$, then it can be shown that

$$\Delta\bar{J} = \underline{\lambda}^T(t_o)\Delta\underline{x}(t_o) \tag{5-13}$$

Thus the effect of small perturbations $\Delta\underline{x}(t_o)$ on $\bar{J}$ can be investigated for $\underline{x}_{DS}$ by considering the sign of $\Delta\bar{J}$. Further, assuming bore-sight at t_f, $X(\underline{x}(t_f)) = 0$ and thus

$$\bar{J} = \phi(\underline{x}(t_f)) \tag{5-14}$$

Since $\phi = 1-P_K$ (from Eq (4-4)), it follows that a positive $\Delta\bar{J}$ represents a reduction in $P_K(t_f)$.

Table 1 below shows the approximate values of $\underline{\lambda}(t_o)$ for the two trajectories which intersect to give the point $\underline{x}_{DS}$.

Table I

Values of $\underline{\lambda}_{DS}(t_o)$ at $\underline{x}_{DS}$

	Trajectory 1 $(q(t_f) > 0)$	Trajectory 2 $(q(t_f) < 0)$
$\lambda_1(t_o)$	-.00046	-.00046
$\lambda_2(t_o)$	+.00044	+.00044
$\lambda_3(t_o)$	+.00001	-.00001

Three observations may be made about the behavior of $\Delta\bar{J}$ due to arbitrary

$\Delta\underline{x}(t_o)$:

(a) Negative Δx_1 always result in a reduction in $P_K(t_f)$ for constant x_2 and x_3. Thus, by definition, E escapes.

(b) Positive Δx_1 always results in an increase in $P_K(t_f)$, i.e., P captures.

(c) Either $\pm\ \Delta x_2$ or $\pm\ \Delta x_3$ result in a reduction in $P_K(t_f)$ for the two trajectories, thus relieving E of his dilemma over choice of strategy at t_o, and enabling escape.

The conclusion made is that the point $\underline{x}_{DS}$ is one bound for the starting points on the p-axis from which a $P_K(t_f) = 5/9$ can be achieved. As such, $\underline{x}_{DS}$ provides a partition of the game space.

Obviously, the starting points discussed above are also bounded at some point on the positive p-axis. Physical argument indicates that this point is the point on the axis where $P_K = 5/9$ and P is boresighted on E. This point is labelled $\underline{x}_{PB}$ in Figure 8. It is readily deduced from the dynamics of the game that P cannot capture from a point nearer to E on the p-axis than $\underline{x}_{PB}$.

The overall conclusion made is that, at least for starting points on the p-axis, the game space can be partitioned into sets of points which represent escape and capture regions.

Partition Diagrams

The optimal trajectories developed for the constant velocity model are 3-dimensional, the variables being the two position coordinates and the relative heading. This raises the problem of presentation. Two methods appear to be available:

(a) Projection of the trajectories on to one plane of the space.

(b) Parameterization of one of the variables.

Both of the possibilities offer advantages. Using one variable as a parameter allows the neutral starting points to be represented as a two dimensional curve enclosing the capture space. Projection permits the capture space to be diagrammed as bounds on two of the variables. Both methods are considered.

The Parameter Method. Partition of the values of p has been shown for those starting points on the axis co-linear with $\underline{V}_e$ (the p-axis). Effectively, this represents a parameterization in q for q=0. Characteristically, each optimal trajectory obtained for the game intersects the p-axis (q=0) at particular values of p and (x_3-x_4). The perturbation analysis previously developed can be used to show that the relative heading at intersection is critical, and thus represents a bound on the pursuer's heading for termination at $P_K(t_f) = 5/9$. For q=0, the game space can thus be divided into escape and capture regions as shown in Figure 9 overleaf.

The Projection Method. The obvious plane on which to protect the capture region is the p-q plane, since this represents a simple physical interpretation. To fully define the capture/escape regions, the projections would enclose all the positions (p,q) from which P could attain capture given that his heading was sufficiently advantageous. The bound on the region would be those points where P had the most advantageous heading and could achieve a maximum $P_K(t_f)$ of 5/9.

The methods developed here do not include a simple way of obtaining a complete partition of the game space. Each value of relative heading requires consideration and the bound on position must be established for each. So far, it appears that this can only be achieved by a long and tedious analysis of the optimal trajectories obtained from all admissable

36

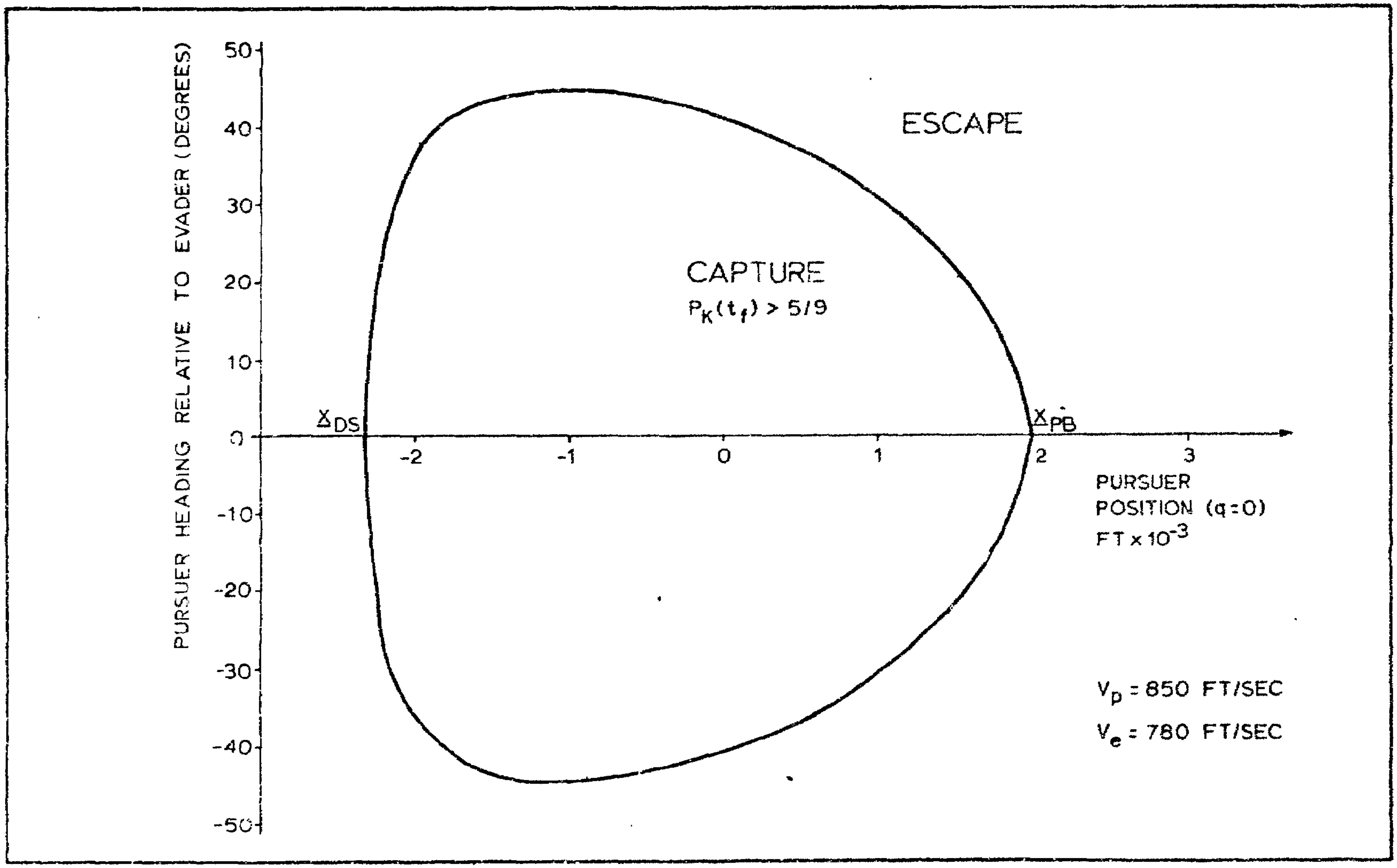

Fig. 9. Escape and Capture Regions (Parameter q=0)

end points.

However, it seems from the results obtained that a reasonable approximation can be made. Analysis of the available trajectories reveals that the singular arc trajectory "encloses" (in the positional sense) all other trajectories except those resulting from a switch in u_p on the singular arc itself. An approximate partition made on this basis is shown in Figure 10. The escape region could be considered as the positions from which P cannot capture regardless of heading. It should be remarked that this partition is not mathematically justified, but represents the author's interpretation of the optimal trajectories studied.

Summary

This chapter contains the major part of the analysis of the problem of defining the escape and capture regions for the pursuit-evasion game studied. The analysis was achieved by a largely experimental means, that is, by physical examination of the optimal trajectories generated. Mathematical deduction enabled specific definition in some cases, but in general no compact analytical method could be developed. However, there appears to be sufficient evidence to suggest that the game space for this class of game can be partitioned into escape and capture regions.

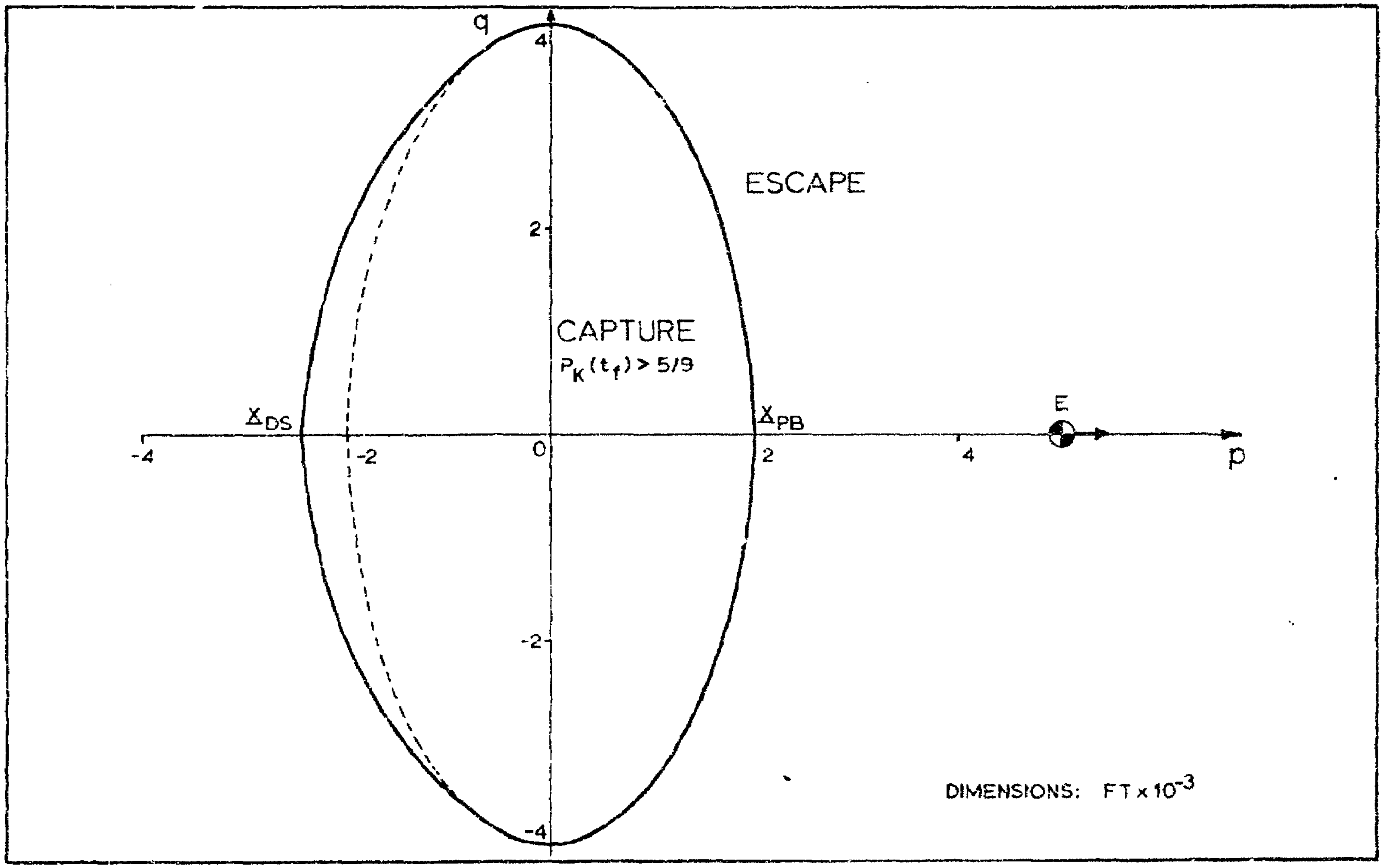

Fig. 10. Approximate Projection of Partition on the p-q Plane

VI. Analysis of the Variable Velocity Model

Chapter V dealt with the major part of the analysis done on the pursuit-evasion problem treated in the thesis. A certain amount of effort was expended on the variable velocity model, and the purpose of this chapter is to show the formulation and discuss the solutions obtained.

Mathematical Statement

The state and costate differential equations are

$$\dot{x}_1 = x_5\cos x_3 - x_6\cos x_4$$

$$\dot{x}_2 = x_5\sin x_3 - x_6\sin x_4$$

$$\dot{x}_3 = \frac{g}{x_5}\, u_p \qquad (6\text{-}1)$$

$$\dot{x}_4 = \frac{g}{x_6}\, u_e$$

$$\dot{x}_5 = A_1 + A_2x_5 + A_3x_5{}^2 + \frac{A_4}{x_5{}^2}\,(u_p{}^2 + 1) + \frac{A_5}{x_5{}^6}\,(u_p{}^2 + 1)^2$$

$$\dot{x}_6 = A_1 + A_2x_6 + A_3x_6{}^2 + \frac{A_4}{x_6{}^2}\,(u_e{}^2 + 1) + \frac{A_5}{x_6{}^6}\,(u_e{}^2 + 1)^2$$

$$\dot{\lambda}_1 = 0$$

$$\dot{\lambda}_2 = 0$$

$$\dot{\lambda}_3 = x_5(\lambda_1\sin x_3 - \lambda_2\cos x_3)$$

$$\dot{\lambda}_4 = x_6(-\lambda_1\sin x_4 + \lambda_2\cos x_4) \qquad (6\text{-}2)$$

$$\dot{\lambda}_5 = -\lambda_1\cos x_3 - \lambda_2\sin x_3 + \lambda_3\,\frac{g}{x_5{}^2}\,u_p - \lambda_5[A_2 + 2A_3x_5$$

$$- 2\,\frac{A_4}{x_5{}^3}\,(u_p{}^2 + 1) - \frac{6A_5}{x_5{}^7}\,(u_p{}^2 + 1)^2]$$

$$\dot{\lambda}_6 = \lambda_1 \cos x_4 + \lambda_2 \sin x_4 + \lambda_4 \frac{g}{x_6^2} u_e - \lambda_6 [A_2 + 2a_3 x_6$$

$$- \frac{2A_4}{x_6^3} (u_e^2 + 1) - \frac{6A_5}{x_6^7} (u_e^2 + 1)^2]$$

When the optimal controls are interior to the control constraints, then for the minimax value of H it is necessary that:

$$H^*_{u_p} = 0 \quad \text{and} \quad H^*_{u_e} = 0 \tag{6-3}$$

Application of these conditions gives the two equations:

$$u_p^3 + \left(\frac{A_4 x_5^4}{2A_5} + 1 \right) u_p - \frac{\lambda_3}{\lambda_5} \frac{g\ x_5^5}{4A_5} = 0 \tag{6-4}$$

$$u_e^3 + \left(\frac{A_4 x_6^4}{2A_5} + 1 \right) u_e - \frac{\lambda_4}{\lambda_6} \frac{g\ x_6^5}{4A_5} = 0 \tag{6-5}$$

Examination of the coefficients of Eqs (6-4) and (6-5) shows that the equations have one real root. Numerical solution yields u_p^* and u_e^*.

The minimax value of H may be verified by the sufficient conditions:

$$H^*_{u_p u_p} > 0 \quad \text{and} \quad H^*_{u_e u_e} < 0 \tag{6-6}$$

If the optimal controls are on the respective constraint boundaries, then the minimax of H is obtained by the direct application of Eq (3-7).

Trajectory Analysis

To enable comparison with the constant velocity results the same end point conditions were used, i.e.,

$$x_4(t_f) = 0$$

$$x_5(t_f) = 850 \text{ ft/sec}$$

$$x_6(t_f) = 780 \text{ ft/sec}$$

v variable

A number of optimal trajectories were obtained and subjected to a similar analysis to that used in the constant velocity model. The trajectories obtained behaved in very much the same manner as for the previous model. However, the same conclusions are difficult to draw because of the introduction of variable x_5 and x_6 (V_p and V_e) which adds two dimensions to the solution trajectories. These are now 5-dimensional, as would be the resulting capture/escape regions of the game space.

The problem introduced by the added dimensions proved insuperable. When the analysis was commenced, it was hoped that the aircraft velocities on the solution trajectories would remain largely constant. This did not prove to be the case, and no acceptable method could be found of handling the added dimensions diagrammatically. However, some general conclusions can be made.

Dispersal Surfaces. The existence of a dispersal surface for the variable velocity model can be demonstrated by similar reasoning to that employed previously in Chapter V. It also appears from observation of the solution trajectories that the point on the dispersal surface on the p-axis provides a bound on starting points on this axis. However, even when q is set to zero in this way, the starting points from intersections on the p-axis are 4-dimensional, and consequently difficult to represent by diagrams.

Capture/Escape Regions. The trajectory which yields the point ($\underline{x}_{DS}$) on the dispersal surface also appears to play an important role in defining escape and capture regions. In the projection on the p-q plane, the trajectory yielding $\underline{x}_{DS}$ appears to "enclose" most of the trajectories which have the required ending. It does not, as such, constitute a mathematical partition of the game space, but it does provide an indication

of the size and shape of the projection of the partition. This in turn gives an indication of the capability of the pursuer to achieve capture. Figure 11 overleaf shows the trajectory which results in $\underline{x}_{DS}$ for the present case. As before, the position coordinates x_1 and x_2 are transformed to the p-q frame by using Eqs (4-7) and (4-8).

Summary

The results achieved for the variable velocity model do not enable the definition of escape and capture regions for this model. However, some contributions are made. The presence of the dispersal surface is demonstrated, and the trajectories obtained give an indication of the pursuer's capability against a maneuvering opponent.

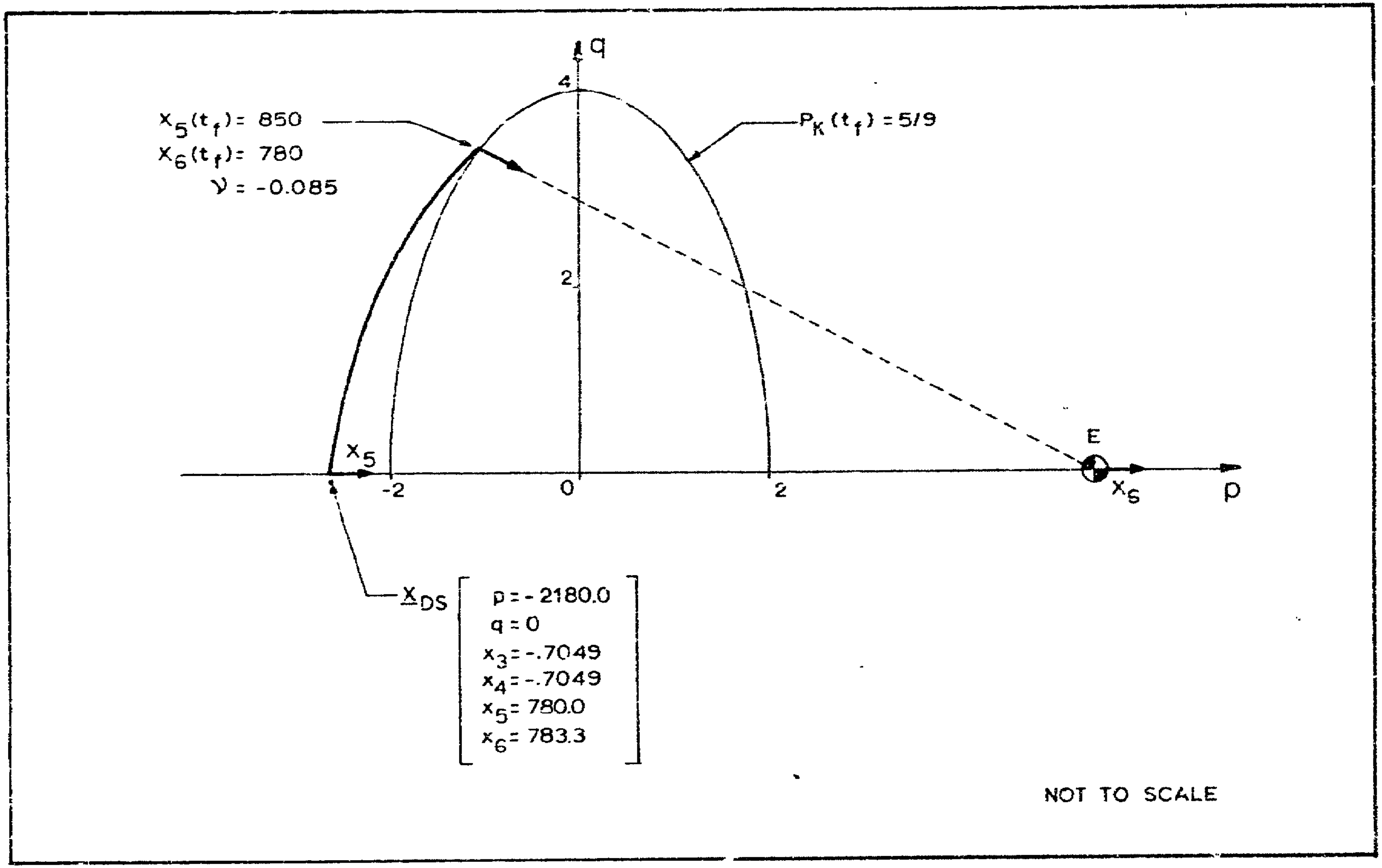

Fig. 11. Dispersal Surface Location for the Variable Velocity Model

VII. Discussion of Results

The purpose of this chapter is to discuss the results obtained, and to consider some of the possible implications of the methods developed in the thesis. Also considered is the extent to which the thesis objectives were achieved.

Thesis Objectives

The thesis objectives were threefold:

(a) To develop a method of defining escape and capture regions for a two aircraft pursuit-evasion game.

(b) To consider the dependence of these regions on the variables in the problem.

A4

(c) To consider the impact that definition of the regions might have on tactics and design.

The second two objectives are, of course, dependent on the first; these are discussed subsequently.

It is felt that the primary objective has only partially been attained. Partition of the game space into escape and capture regions was achieved in only certain cases for the most simple model. However, the methods developed appear to offer possibilities to future researchers, and represent at least the initial steps toward complete solution of the problem.

Analysis Methods

The methods used here to analyze solution trajectories are largely empirical. That is, many trajectories were generated and conclusions made from their physical behavior. It does not seem that any concise analytical method exists which would reduce this physical effort. A

second major problem, previously mentioned, concerns the dimensions of the game space and the solution trajectories. The inability of conventional graphical techniques to handle surfaces of 4 or more dimensions handicaps the analysis of solutions. Removal of this restraint would also ease the difficulties encountered in presentation of the solutions.

The most basic ingredient of the methods used was to specify the sets of admissable end points for the differential game. While this results in many trajectories which may be tedious to analyze, it avoids the necessity of solving a two point boundary value problem (TPBVP). The TPBVP in a differential game can be extremely difficult to solve because of the iterative nature of solution methods. Also, the solution provides information about only one trajectory, some portions of which may be non-optimal due to the presence of singular surfaces in the game space. In contrast, provided admissable end points can be specified, backward integration is a very speedy and simple method of obtaining many solution trajectories. This type of approach may well have applications in other fields of optimal control and differential games.

Variation of Game Parameters

Some effort was made to evaluate the dependence of the partitions of the space on the game parameters, but insufficient analysis was achieved to enable specific conclusions to be made. The most significant parameter is considered to be the distance that the P_K region is from E. At maximum load factor of 5, E's turn rate is about 0.2 rads/sec, which means that the P_K = 1.0 point has an angular velocity of 1000 ft/sec. This single factor contributes greatly to E's ability to escape. Reducing V_e causes E to be lift-limited in flight (Appendix A); at the velocity

used in illustration (780 ft/sec) E's turn rate is still about 0.18 rads/sec.

Some experiment was made in two areas for the constant velocity model:

(a) Increasing P's velocity advantage over E

(b) Arbitrarily reducing E's turning capability ($u_{e_{max}}$).

Both these variations produced results analogous to the case analyzed in Chapter V with the expected increase in the sizes of the capture regions. However, beyond a certain reduction in $u_{e_{max}}$, the behavior of the solutions changed. The envelope of admissable end points reduces in size, and no singular arc trajectories can be found. In addition, those trajectories examined did not appear to enclose the capture space in the same manner as before. The value of $u_{e_{max}}$ at which most investigation was done was equivalent to a maximum load factor of 2.5.

Insufficient investigation was achieved to allow specific conclusions, but the following observations are made:

(a) It is probable that solution trajectories are non-unique; that is, two or more trajectories have the same end point. The analysis of numerical solutions is then invalid.

(b) Consideration of solutions for low values of P_K (e.g., $P_K = 5/9$) may be misleading when E's turning capability is drastically reduced. This is because P may be able to excede the selected $P_K(t_f)$ easily from a great variety of "advantageous" positions. Solutions with a low value of $P_K(t_f)$ may then only have significance if they arise from initial conditions at which P is at a disadvantage relative to E.

Tactical and Design Implications

As previously stated, the simplicity of the aircraft models used in

the thesis limit the conclusions that may be drawn about real air combat situations. However, there are inferences which are considered to have some validity.

The most obvious conclusion is that the capture regions are extremely small for two-dimensional maneuver. The single most important factor contributing to this is the angular velocity which the P_K, regions move as a result of the evader's turn rate. In reality, however, the situation is not quite as adverse as it appears, since current analysis shows that the pursuer does in fact have some opportunity for a "side-shot." That is, in turning, the evader opens up a P_K region on the inside of the turn. This was not modeled for this investigation, and represents an area where further fruitful research might be accomplished.

Also it is difficult to logically extend the results of this investigation to three dimensions. One observation is that the results obtained here are valid for motion in any plane in the absence of gravity. An inference from this is that it is difficult to see how the pursuer's capability could be improved in three-dimensional maneuver. Specific results in this area are once again subject to a great deal more research.

From the tactical view point, it is felt that there is sufficient evidence from this research to indicate severe limitations on the use of this particular airborne missile system. While its performance against an unsuspecting target may be adequate, evasive maneuvers on the part of the evader cause a great reduction in the pursuer's capability.

In the design field, the approach presented here is felt to have considerable potential. This potential lies not only in evaluating the capability of a given system, but also in the comparison of different

weapons systems. Realization of the potential is dependent upon several factors. A refinement of the general approach developed is necessary, and the obstacle of graphically displaying multi-dimensional surfaces must be overcome. Of the two, the latter presents the greatest problem, and solution would greatly benefit future research.

Summary

In general, the research presented here has met with limited success in relation to the overall objectives. However, several original solution methods are employed, and the development potential offers many opportunities for future research.

VIII. Conclusions and Recommendations

Conclusions

A two aircraft pursuit-evasion game has been posed as a free time, zero sum, perfect information differential game. A method of determining admissable end points to the game was developed, permitting the use of numerical backward integration techniques to produce optimal trajectories.

Two aircraft models were used, in both of which motion was restricted to level flight at constant altitude. The simpler model had the additional restraint that both aircraft move with constant velocity. Examination of the optimal trajectories obtained yielded a partition of the game space into escape and capture regions under certain conditions for the simple model. Complete partition of the game space was not achieved for either model, but the existence of partition is shown, and a general approximation made within the limits of the graphical techniques available.

From the practical standpoint, it can be concluded that a weapon system with characteristics similar to those considered in the present case may have severe limitations when faced with intelligent opposition. Although the constraints imposed in this analysis are unrealistic in some senses, there is sufficient evidence to make the preceding conclusion.

In terms of evaluating the capability of air-to-air weapon systems, the methods developed here are considered to have significant potential; this potential is dependent upon further refinement of the techniques employed.

Recommendations

The methods developed in this thesis represent the first steps towards an analytical solution to the problem of defining escape and

capture regions for aerial pursuit-evasion games. The greatest hurdle encountered was the graphical limitations on presentation. In the event that further research is attempted, the following areas are recommended:

(a) Refinement and generalization of the analysis of optimal trajectories generated by backward integration.

(b) Determination of graphical methods for demonstrating multi-dimensional surfaces and trajectories.

Bibliography

1. Anderson, Gerald M. Necessary Conditions for Singular Solutions in Differential Games with Controls Appearing Linearly. Proceedings of the First International Conference on the Theory and Application of Differential Games. Amherst, Massachusetts: University of Massachusetts, 1969.

2. Bryson, A. E. and Y. C. Ho. Applied Optimal Control. Waltham, Massachusetts: Blaisdell Publishing Company, 1969.

3. Isaacs, R. Differential Games. New York: John Wiley and Sons, Inc., 1965.

4. McDonnell Douglas Aircraft Corporation. F4-E Performance Data and Substantiation. F696 V.1. St. Louis:

51

Appendix A

Development of the Aircraft Model

Purpose

The purpose of this appendix is to show the development of the aircraft dynamic model used in the thesis. The model is generalized to 3 dimensions and then specialized to the final planar model used.

Aircraft Dynamics

The model is based upon empirical data for the F4-E published in Reference (4). For each of 3 altitudes analytic functions are developed for thrust (T) and lift limit. The coefficient of drag (C_D) is related analytically to lift coefficient (C_L) for subsonic flight with validity up to a Mach No (M) of 0.9.

Thrust (T). Both aircraft are assumed to be using full after-burner thrust during combat. Figure 12 overleaf shows the variation of thrust with velocity and altitude. A linear approximation to the curves over the velocity range of interest yields

$$T = a + bh + cV \tag{A-1}$$

where

$$a = 22{,}346.7 \text{ lb}$$

$$b = 0.7018 \quad \text{lb/ft}$$

$$c = 18.141 \quad \text{lb/ft/sec}$$

For a constant altitude of 20,000 ft. the relation becomes

$$T = 8310.7 + 18.141V \tag{A-2}$$

Lift Limit. The maximum lift that can be developed for a given velocity is

$$L = \frac{1}{2}\rho V^2 S C_{L\max} \tag{A-3}$$

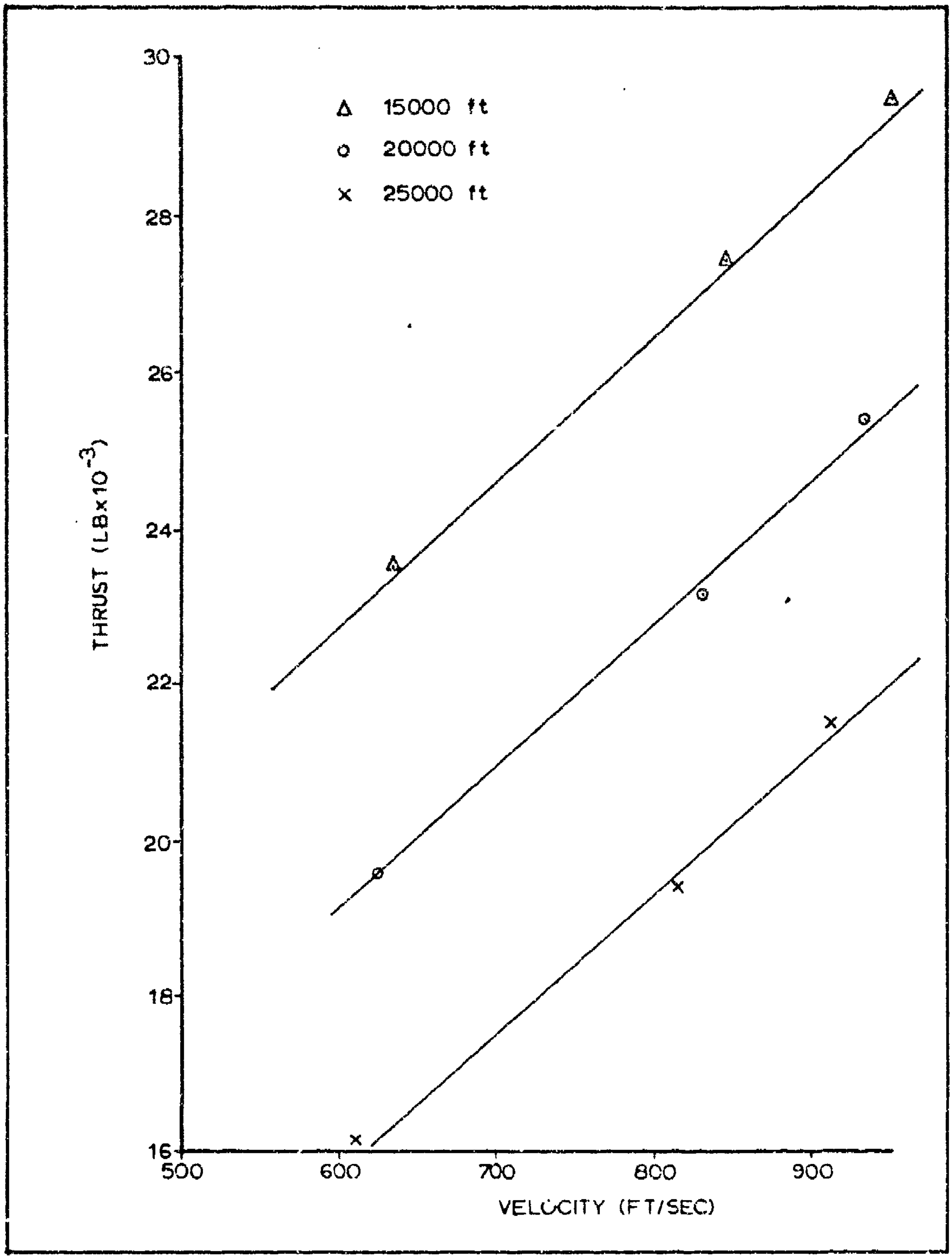

Fig. 12. Variation of Thrust with Velocity

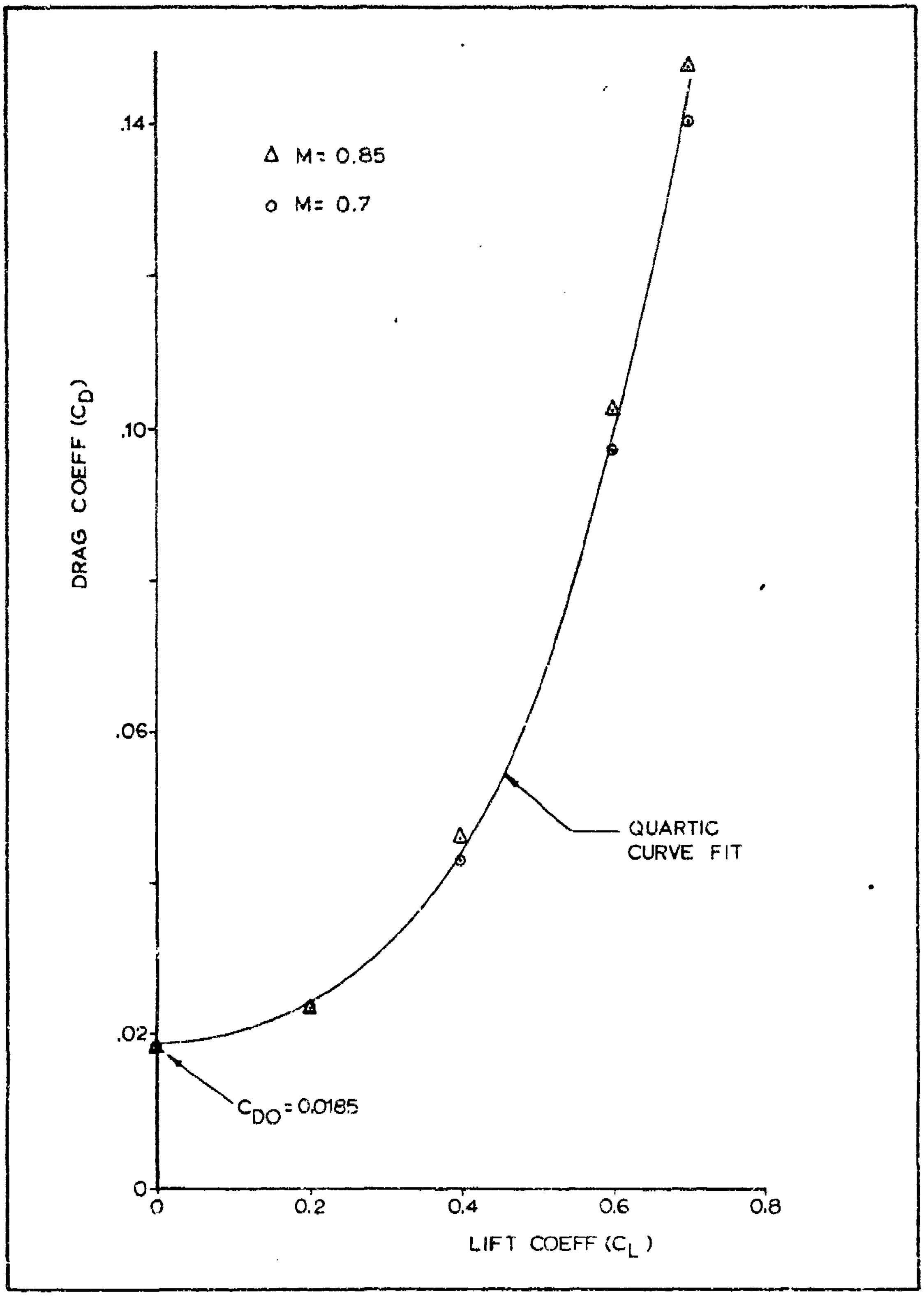

Fig. 13. Variation of C_D with C_L

and load factor (n) is defined as

$$n = \frac{L}{W} \tag{A-4}$$

hence the maximum load factor attainable is

$$n_{max} = \frac{1}{2} \frac{\rho V^2 S}{W} C_{Lmax} \tag{A-5}$$

For flight regimes where n is not limited by Eq (A-5), the maximum value of n is taken as 5. This represents a realistic limit imposed by a pilot's capability to withstand sustained acceleration forces. Lift limited load factor is plotted against velocity in Figure 14 (Page 56) with altitude as a parameter. A linear fit gives

55

$$n_{max} = 0.16 + (0.01422 - .000304h)\ (V - 250) \tag{A-6}$$

For planar flight at 20,000 ft

$$n_{max} = 01.875 + .00814V \tag{A-7}$$

<u>Drag Coefficient</u>. C_D is graphed in Figure 13 (Page 54) as a function of C_L. A quartic curve fit is made, giving

$$C_D = C_{DO} + k_1\ C_L{}^2 + k_2 C_L{}^4 \tag{A-8}$$

where

$$C_{DO} = 0.0185$$

$$k_1 = 0.1007$$

$$k_2 = 0.3261$$

<u>Equations of Motion</u>

Figure 15 (Page 57) shows diagramatically the variables used to describe the aircraft states in 3 dimensions. The equations of motion may be written

$$\dot{x} = V\cos\gamma\cos\theta$$

$$\dot{y} = V\cos\gamma\sin\theta$$

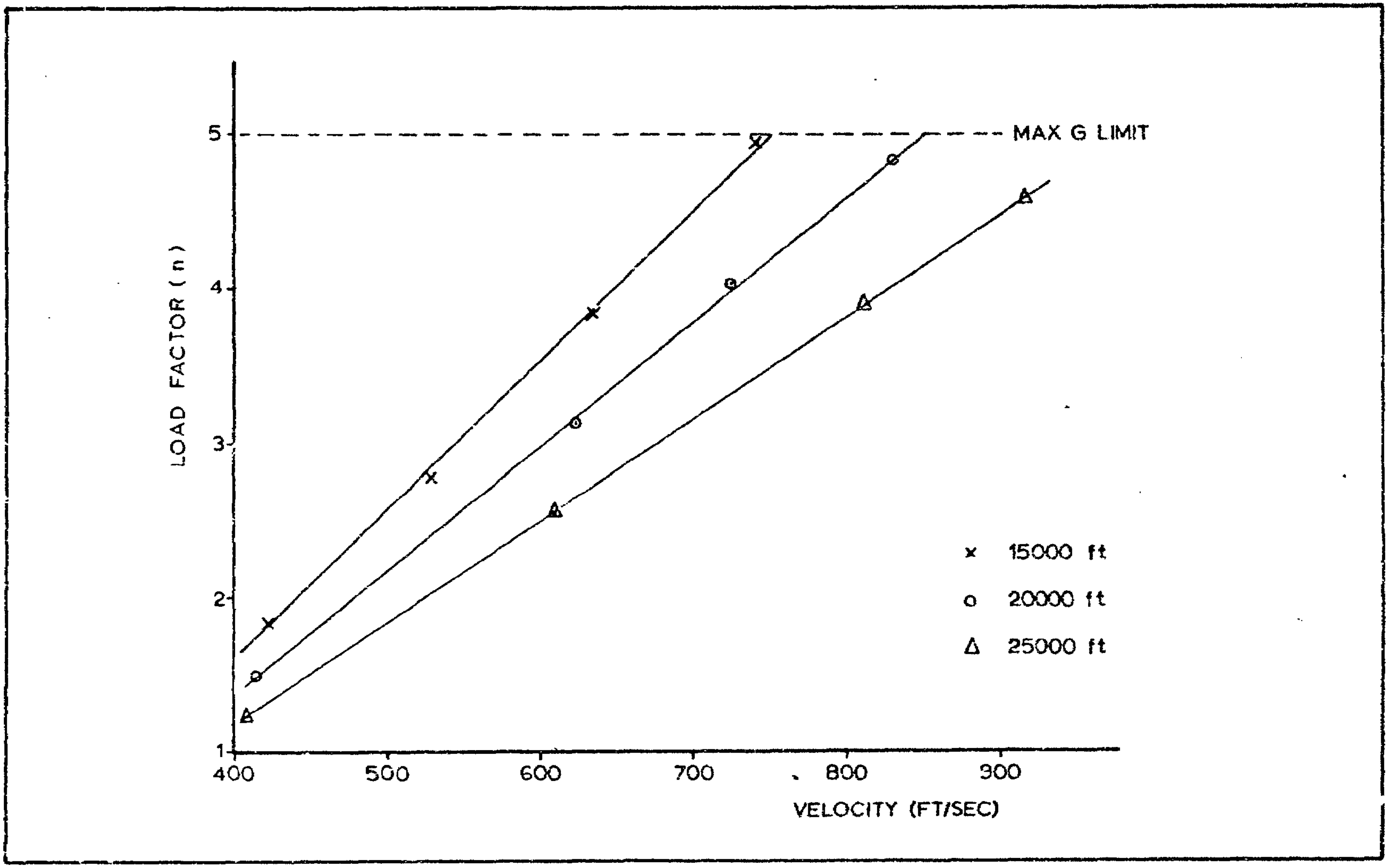

Fig. 14. Load Factor for Lift Limited Flight

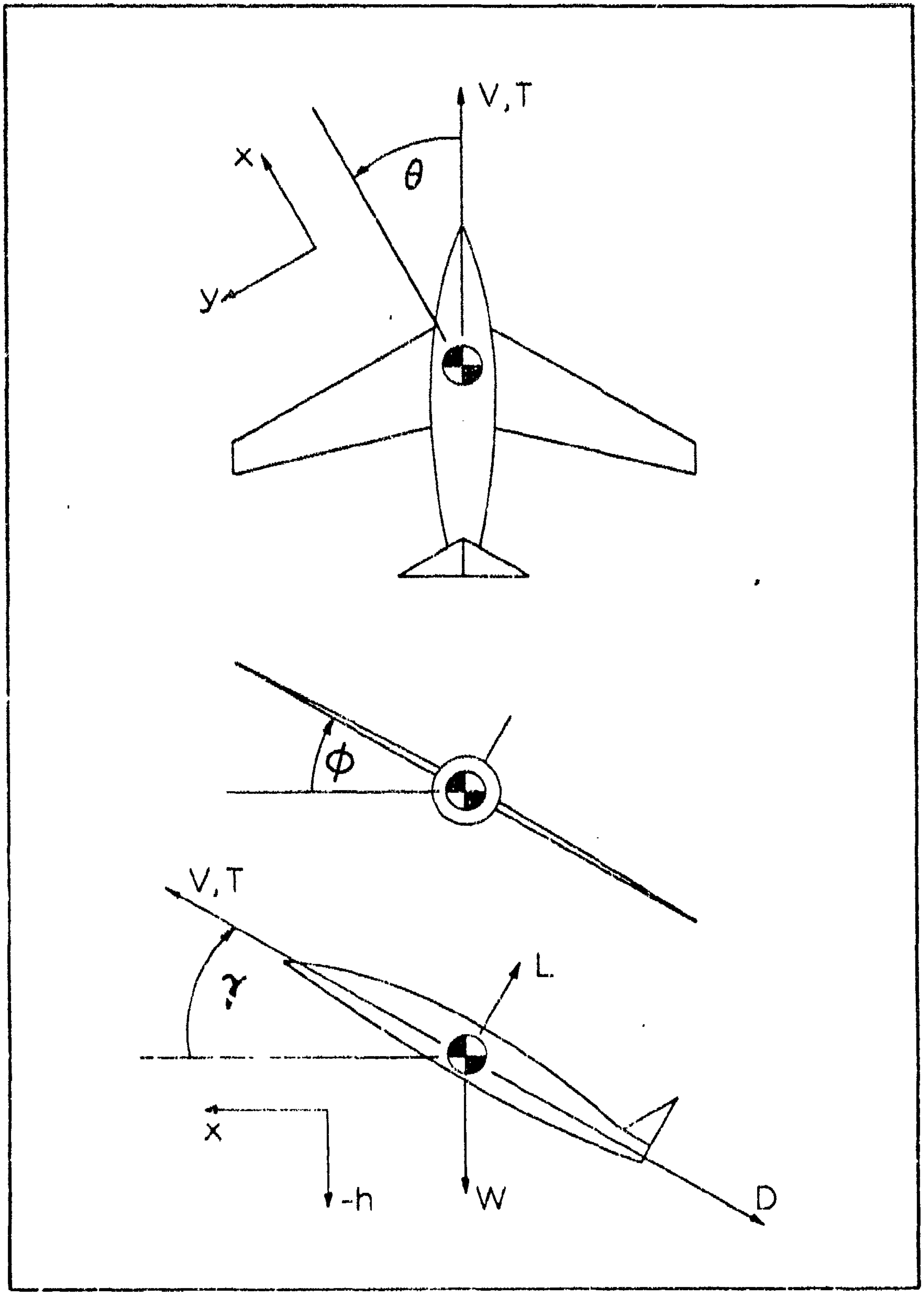

Fig. 15. Definition of Aircraft Variables

$$\dot{h} = V\sin\gamma$$

$$\dot{\theta} = \frac{gL\sin\phi}{WV\cos\gamma} \qquad \text{(A-9)}$$

$$\dot{\gamma} = \frac{g}{V}\left(\frac{L\cos\phi}{W} - \cos\gamma\right)$$

$$\dot{V} = \frac{g}{W}(T - D)$$

Constant altitude flight requires that

$$L\cos\phi = W \qquad \text{(A-10)}$$

and hence

$$n = \frac{L}{W} = \frac{1}{\cos\phi} \qquad \text{(A-11)}$$

thus

$$\sin\phi = \pm\frac{1}{n}(n^2-1)^{1/2} \qquad \text{(A-12)}$$

If the control, u, is defined by

$$u = \pm(n^2-1)^{1/2} \qquad \text{(A-13)}$$

then for constant altitude flight, Eq (A-9) can be written

$$\dot{x} = V\cos\theta$$

$$\dot{y} = V\sin\theta \qquad \text{(A-14)}$$

$$\dot{\theta} = \frac{gu}{V}$$

$$\dot{V} = \frac{g}{W}(T - D)$$

The limits on u such that

$$u_{min} \leq u \leq u_{max} \qquad \text{(A-15)}$$

are imposed by the limits on load factor n.

The drag is given by

$$D = \frac{1}{2}\rho V^2 S(C_{DO} + k_1 C_L{}^2 + k_2 C_L{}^4) \qquad \text{(A-16)}$$

and

$$C_L = \frac{L}{\frac{1}{2}\rho V^2 S} \tag{A-17}$$

thus

$$D = A_3V^2 + A_4\frac{(u^2+1)}{V^2} + \frac{A_5(u^2+1)^2}{V^6} \tag{A-18}$$

where

$$A_3 = \frac{1}{2}\rho S C_{DO} \tag{A-19}$$

$$A_4 = \frac{2k_1W^2}{\rho S} \tag{A-20}$$

$$A_5 = \frac{2k_2W^4}{\rho S} \tag{A-21}$$

59

From Eq (A-2), thrust can be written

$$T = A_1 + A_2V \tag{A-22}$$

Using Eqs (A-18) and (A-22), equations of motion become

$$\dot{x} = V\cos\theta$$

$$\dot{y} = V\sin\theta$$

$$\dot{\theta} = \frac{gu}{V}$$

$$\dot{V} = A_1 + A_2V + A_3V^2 + \frac{A_4(u^2+1)}{V^2} + \frac{A_5(u^2+1)^2}{V^6} \tag{A-23}$$

For constant velocity flight $\dot{V} = 0$.

Appendix B

The Necessary and Junction Conditions for Singular Arcs

Purpose

The purpose of this appendix is to summarize the necessary and junction conditions for singular arcs in the optimal solutions to a differential game where the players' controls appear linearly in the Hamiltonian. The summary is based upon the derivation developed in Ref (1).

Problem Formulation

The state equations are

$$\dot{\underline{x}} = \underline{f}(\underline{x},u,v) \tag{B-1}$$

and $\underline{f}$ is linear in u and v. The Hamiltonian (H) is

$$H = \underline{\lambda}^T\underline{f} \tag{B-2}$$

where $\underline{x}$ is the n-dimensional state vector, and $\underline{\lambda}$ the n-dimensional co-state vector, subject to

$$\dot{\underline{\lambda}} = - H_{\underline{x}} \tag{B-3}$$

The terminal values $\underline{x}(t_f)$ and $\underline{\lambda}(t_f)$ are assumed specified.

A payoff function J is specified and the objective of the game is to find u* and v* for the saddle point solution

$$\min_u \max_v J\ (u^*,\ v^*) = \max_v \min_u J\ (u^*,\ v^*) \tag{B-4}$$

over the time internal $[t_0,\ t_f]$. The controls u and v are subject to the constraints

$$|u| \leq u_{max}\ ,\quad |v| \leq v_{max} \tag{B-5}$$

Necessary Conditions

For a saddle point solution in J, a saddle point in H is necessary such that

$$\min_{u} \max_{v} H = \max_{v} \min_{u} H \tag{B-6}$$

If switching functions S_u and S_v are defined such that

$$S_u(\underline{x},\underline{\lambda}) = \frac{\partial H}{\partial u} \tag{B-7}$$

$$S_v(\underline{x},\underline{\lambda}) = \frac{\partial H}{\partial V} \tag{B-8}$$

then the saddle point controls are given by

$$\begin{aligned} u &= u_{max} \quad \text{if} \quad S_u < 0 \\ u &= u_{min} \quad \text{if} \quad S_u > 0 \end{aligned} \tag{B-9}$$

$$\begin{aligned} v &= v_{max} \quad \text{if} \quad S_v > 0 \\ v &= v_{min} \quad \text{if} \quad S_v < 0 \end{aligned} \tag{B-10}$$

Singular Solutions

Due to the linearity in u and v, the Hamiltonian may be written

$$H = \underline{\lambda}^T \underline{g}(\underline{x}) + S_u u + S_v v \tag{B-11}$$

Assume that $S_u = 0$, so that H becomes independent of u; thus minimization of H with respect to u is not possible.

Necessary and Junction Conditions. It can be shown that along a singular arc, in general

$$S_u(\underline{x},\underline{\lambda}) = \dot{S}_u(\underline{x},\underline{\lambda}) = \ldots . \ S_u^{(2q-1)}(\underline{x},\underline{\lambda}) = 0 \tag{B-12}$$

where successive differentiation yields a function which is explicit in u

$$S_u^{(2q)}(\underline{x},\underline{\lambda},u) = 0 \tag{B-14}$$

Eq (B-14) may be solved to yield the minimizing u* on the singular arc, and Eqs (B-13) must hold across the junctions between singular and non-singular arcs.

One further necessary condition is that

$$\frac{\partial}{\partial u} [S_u^{(2q)}] < 0 \qquad \text{(B-15)}$$

where q is odd.

Appendix C

Development of Influence Functions for a Differential Game

Purpose

The purpose of this appendix is to show how the costates ($\underline{\lambda}(t)$) can be used as influence functions to determine the effect of small perturbations ($\Delta\underline{x}(t)$) on an optimal trajectory for a class of differential games.

Mathematical Development

A differential game is assumed with augmented objective function

$$\bar{J} = \phi(\underline{x}(t_f)) + \nu\chi(\underline{x}(t_f)) \tag{C-1}$$

subject to the differential constraints

$$\dot{\underline{x}} = \underline{f}(\underline{x},u,v) \tag{C-2}$$

where the controls, u and v are bounded by

$$u_{min} \leq u \leq u_{max} \tag{C-3}$$

$$v_{min} \leq v \leq v_{max} \tag{C-4}$$

The Hamiltonian, H is formed such that

$$H = \underline{\lambda}^T \underline{f} \tag{C-5}$$

whence, application of the necessary conditions for optimality yields

$$\dot{\underline{\lambda}} = -f_{\underline{x}}^T \underline{\lambda} \tag{C-6}$$

with the end conditions

$$\underline{\lambda}(t_f) = \bar{J}_{\underline{x}} \tag{C-7}$$

making a first order expansion of Eqs (C-2) and (C-6) at some time, t

$$\Delta\dot{\underline{x}} = \underline{f}_{\underline{x}}^T \Delta\underline{x} + \underline{f}_u \Delta u + \underline{f}_v \Delta v \tag{C-8}$$

$$\Delta\dot{\lambda} = -\, f_{\underline{x}}^{T}\, \Delta\underline{\lambda} \tag{C-9}$$

Combining Eqs (C-8) and (C-9)

$$\begin{aligned} \underline{\lambda}^T\, \Delta\dot{\underline{x}} + \dot{\underline{\lambda}}^T\, \Delta\underline{x} &= \underline{\lambda}^T(\underline{f}_{\underline{x}}^T\, \Delta\underline{x}) + \underline{\lambda}^T\underline{f}_u\, \Delta u + \underline{\lambda}^T\underline{f}_v\, \Delta v \\ &\quad - (\underline{\lambda}^T f_{\underline{x}})\Delta\underline{x} \\ &= \underline{\lambda}^T\underline{f}_u \Delta u + \underline{\lambda}^T\underline{f}_v \Delta v \end{aligned} \tag{C-10}$$

On an optimal trajectory, the necessary conditions following must be satisfied if u and v are unconstrained

$$H_u = 0\ ,\quad H_v = 0 \tag{C-11}$$

Hence from Eq (C-5)

$$\underline{f}_u = 0\ ,\quad \underline{f}_v = 0 \tag{C-12}$$

If u and v are on the constraint boundaries then

$$\Delta u = 0\ ,\quad \Delta v = 0 \tag{C-13}$$

Using Eqs (C-12) and (C-13) in Eq (C-10), then

$$\underline{\lambda}^T\, \Delta\dot{\underline{x}} + \dot{\underline{\lambda}}^T\, \Delta\underline{x} = 0 \tag{C-14}$$

and hence

$$\int_{t_o}^{t_f} (\underline{\lambda}^T\Delta\dot{\underline{x}} + \dot{\underline{\lambda}}^T\Delta\underline{x})dt = \int_{t_o}^{t_f} [\, \frac{d}{dt}\, (\underline{\lambda}^T\Delta\underline{x})]dt = 0 \tag{C-15}$$

$$\therefore\ (\underline{\lambda}^T\Delta\underline{x})\big|_{t_f} - (\underline{\lambda}^T\Delta\underline{x})\big|_{t_o} = 0 \tag{C-16}$$

$$\therefore\ (\underline{\lambda}^T\Delta\underline{x})\big|_{t_f} = (\underline{\lambda}^T\Delta\underline{x})\big|_{t_o} \tag{C-17}$$

Now, from Eq (C-1)

$$\Delta\bar{J} = (\bar{J}_{\underline{x}}\ \Delta\underline{x})\big|_{t_f} \tag{C-18}$$

Thus, combining Eqs (C-7) and (C-18)

$$\Delta\bar{J} = (\underline{\lambda}^T\ \Delta\underline{x})\big|_{t_f} \tag{C-19}$$

The implication of Eqs (C-17) and (C-19) is that

$$\Delta\bar{J} = (\underline{\lambda}^T \, \Delta\underline{x})\big|_{t_o} \tag{C-20}$$

Thus, it can be shown that the costates at t_o ($\underline{\lambda}(t_o)$) are influence coefficients on $\Delta\bar{J}$ at t_f, allowing the determination of the effects of some $\Delta\underline{x}(t_o)$ on the objective function, $\bar{J}$.

Vita

Paddy Cawdery was born on 18 October 1943 in Kisumu, Kenya. He attended the Duke of York School, Nairobi, Kenya from 1956 to 1961. In September 1962, he joined the Royal Air Force and served for a year at the RAF Technical College, Henlow, before going up to the University of Nottingham to read for a Bachelor of Science degree in Electrical Engineering. He graduated in 1966.

He served a tour in Training Command on ground communications and radar, followed by two years at Site III of the Ballistic Missile Early Warning System, RAF Fylingdales, England. The latter tour immediately preceded his posting as an exchange officer with the USAF on the graduate Astronautics course at the Air Force Institute of Technology at Wright-Patterson Air Force Base, Ohio.

This thesis was typed by Miss Sherry L. Willman.